

# COMPUTATION OF SOME EXAMPLES OF BROWN'S SPECTRAL MEASURE IN FREE PROBABILITY

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**ABSTRACT.** We use free probability techniques for computing spectra and Brown measures of some non hermitian operators in finite von Neumann algebras. Examples include  $u_n + u_\infty$  where  $u_n$  and  $u_\infty$  are the generators of  $\mathbf{Z}_n$  and  $\mathbf{Z}$  respectively, in the free product  $\mathbf{Z}_n * \mathbf{Z}$ , or elliptic elements, of the form  $S_\alpha + iS_\beta$  where  $S_\alpha$  and  $S_\beta$  are free semi-circular elements of variance  $\alpha$  and  $\beta$ .

## 1. INTRODUCTION

Recently Haagerup and Larsen [HL99] have computed the spectrum and the Brown measure of  $R$ -diagonal elements in a finite von Neumann algebra, in terms of the distribution of its radial part. The purpose of this paper is to apply free probability techniques for computing spectra and Brown measures of some non-hermitian, and non- $R$ -diagonal elements in finite von Neumann algebras, which can be written as a free sum of an  $R$ -diagonal element and an element with arbitrary  $*$ -distribution. Motivations for this study are twofold, on one hand some of these elements appear as transition operators of random walks on groups or semi-groups, see e.g. [dlHRV93a], [dlHRV93b], [BVZ97], here we shall for example treat linear combinations of  $u_n$  and  $u_\infty$ , the generators of  $\mathbf{Z}_n$  and  $\mathbf{Z}$  in  $\mathbf{Z}_n * \mathbf{Z}$  and  $u_2 + v_2 + u_\infty$ . On the other hand random matrix theory has a close connection with free probability (see [VDN92]), but for the moment very little has been done for understanding limit distributions of spectra of non-normal matrices in terms of free probability. For example, the empirical distribution on the eigenvalues of a random matrix with independent identically distributed complex entries, suitably rescaled, converges, with probability one, as its size grows to infinity, to the circular law (the uniform distribution on the unit disk), see [Gir84], [Gir97a], [Bai97], which is the Brown measure for a circular element, in the sense of Voiculescu. It is known that the circular element is the limit in  $*$ -distribution of the above random matrices, but it is not possible to deduce from this the convergence of the empirical distribution on the spectrum (see Lemma 2.1 below).

Another example that we shall consider in this paper is the free sum of an arbitrary element with a circular element. Hopefully, the corresponding Brown measures should represent limit of eigenvalue distributions of random matrices of the form  $A + W$  where  $A$  is a matrix with some limit  $*$ -distribution, and  $W$  is a matrix with independent entries. In addition to the circular element discussed above, this is known to be true for the so-called *elliptic element*, which can be written as  $S_\alpha + iS_\beta$  and whose Brown measure was first computed in [Lar99] by ad-hoc methods. It turns out to be treatable by our method as well. The empirical eigenvalue distribution of its matrix model with Gaussian random matrices is computed in [HP98] and shown to converge to the uniform measure on its spectrum, an ellipse.

However in this paper we shall stick to the purely free probabilistic aspects of the subject, and not touch upon the random matrix problem. We hope to deal with this somewhere else.

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This paper is organized as follows. In section 2 we recall preliminary facts about Brown measures and free probability theory. In section 3 we give a general approach towards the computation of the Brown measure for the sum of an  $R$ -diagonal element with an arbitrary element. We specialize in sections 3 and 4 to the cases where the  $R$ -diagonal element is a Haar unitary or a circular element, respectively. We close with some final remarks in section 5. The pictures of random matrix spectra appearing in various sections of this papers were computed with `GNU octave` and plotted with `gnuplot`; the plots of densities of various Brown measures, which accompany or replace the rather unwieldy density formulae, were computed by `Mathematica`.

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## 2. PRELIMINARIES

**2.1. The Fuglede–Kadison determinant and Brown’s spectral measure.** Let  $\mathcal{M}$  be a finite von Neumann algebra with faithful tracial state  $\tau$  and denote, for invertible  $a \in \mathcal{M}$ ,  $\Delta(a) = e^{\tau(\log|a|)}$  its Fuglede–Kadison determinant (cf. [FK52]). Denoting by  $\mu_x$  the spectral measure for the self-adjoint element  $x \in \mathcal{M}$ , i.e. the unique probability measure on the real line satisfying  $\tau(x^n) = \int t^n d\mu_x(t)$ , we have the following formula for the logarithm of the determinant, which serves as a definition of the determinant in the case where  $a$  is not invertible

$$\log \Delta(a) = \int_{\mathbf{R}} \log t d\mu_{|a|}(t).$$

The function  $\Delta(\lambda - a)$  is a subharmonic function of the complex variable  $\lambda$ , and there is a unique probability measure  $\mu_a$  on  $\mathbf{C}$ , with support on the spectrum of  $a$ , called the *Brown measure* of  $a$ , such that

$$\log \Delta(\lambda - a) = \int \log |\lambda - z| \mu_a(dz);$$

it is given by

$$\mu_a = \frac{1}{2\pi} \nabla^2 \log \Delta(\lambda - a)$$

where  $\nabla^2$  is the Laplace operator in the complex plane, in the sense of distributions (see [Bro86]). If  $a$  is normal, then  $\mu_a$  is just the spectral measure of  $a$ . When  $\mathcal{M}$  is  $M_n(\mathbf{C})$ , with the canonical normalized trace, then  $\mu_a$  is the empirical distribution on the spectrum of  $a$  (counting multiplicities). Although the Brown measure of  $a$  can be computed from its  $*$ -distribution, i.e. the collection of all its  $*$ -moments  $\tau(a^{\varepsilon_1} a^{\varepsilon_2} \cdots a^{\varepsilon_n})$ , where  $a^{\varepsilon_j}$  is either  $a$  or  $a^*$ , it does not depend continuously on these  $*$ -moments. Indeed let for example  $a_n$  be the  $n \times n$  nilpotent matrix with ones on the first upper diagonal and zeros everywhere else, then as  $n$  goes to infinity the  $*$ -moments of  $a_n$  converge towards those of a Haar unitary (a unitary element  $u$  with  $\tau(u^n) = 0$  for  $n \neq 0$ ), whose Brown measure is the Haar measure on the unit circle, whereas the Brown measure of  $a_n$  is  $\delta_0$  for all  $n$ .

**Lemma 2.1.** *Let  $(a_n; n \geq 0)$  be a uniformly bounded sequence whose  $*$ -distributions converge towards that of  $a$ , and suppose the Brown measure of  $a_n$  converges weakly towards some measure  $\mu$ , then one has*

- (i)  $\int \log |\lambda - z| \mu(dz) \leq \Delta(\lambda - a) = \int \log |\lambda - z| \mu_a(dz)$  for all  $\lambda \in \mathbf{C}$
- (ii)  $\int \log |\lambda - z| \mu(dz) = \Delta(\lambda - a) = \int \log |\lambda - z| \mu_a(dz)$  for all  $\lambda$  large enough.

*Proof.* The distribution of  $|\lambda - a_n|$  has a support which remains in a fixed compact set, and it converges weakly towards that of  $|\lambda - a|$ . Part (i) follows from this and the fact that the function  $\log$  is a limit of a decreasing sequence of continuous functions. If  $\lambda$  is large enough, then the union of the supports of the distributions of the  $|\lambda - a_n|$  is away from 0, hence the function  $\log$  is continuous there and (ii) follows from weak convergence.  $\square$

The outcome of (i) of the Lemma is that the measure  $\mu_a$  is a balayée of measure  $\mu$ , while we get from (ii) the following

**Corollary 2.2.** *Let  $U_a$  be the unbounded connected component of the complement of the support of  $\mu_a$ , then the support of  $\mu$  is included in  $\mathbf{C} \setminus U_a$ .*

*Proof.* The function  $\int \log |\lambda - z| \mu_a(dz)$  is harmonic in  $U_a$ , while  $\int \log |\lambda - z| \mu(dz)$  is subharmonic there, consequently  $\int \log |\lambda - z| \mu_a(dz) - \int \log |\lambda - z| \mu(dz)$  is a nonnegative superharmonic function on  $U_a$ . Since this function attains the value 0 by (ii), it is identically 0 by the minimum principle, therefore  $\int \log |\lambda - z| \mu(dz)$  is harmonic on  $U_a$ , and thus the support of  $\mu$  is included in  $\mathbf{C} \setminus U_a$ .  $\square$

Conversely, given two measures  $\mu$  and  $\mu_a$  on  $\mathbf{C}$  satisfying (i) and (ii), we do not know whether there always exists a corresponding sequence  $(a_n)_{n \geq 0}$ , fulfilling the hypotheses of Lemma 2.1

**2.2.  $R$ - and  $S$ -transforms.** We shall refer to [VDN92], and [Voi98] or [HP99] for basic concepts of free probability theory. Let  $(\mathcal{M}, \tau)$  be as in section 2.1, and let  $a \in \mathcal{M}$ . The power series

$$G_a(\zeta) = \frac{1}{\zeta} \sum_{n=0}^{\infty} \frac{\tau(a^n)}{\zeta^n}$$

can be inverted (for composition of formal power series), in the form

$$K_a(z) = \frac{1}{z} + \sum_{n=0}^{\infty} c_{n+1} z^n = \frac{1}{z}(1 + R_a(z)).$$

The power series  $R_a$  is called the *R-transform* of  $a$  (note that this slightly differs from the original definition of Voiculescu) and its coefficients are called the *free cumulants* of  $a$ . Let

$$\psi_a(z) = \sum_{n=1}^{\infty} \tau(a^n) z^n = \frac{1}{z} G_a\left(\frac{1}{z}\right) - 1$$

be the generating moment series for  $a$ , and assume that the first moment is nonzero, so that  $\psi'_a(0) \neq 0$ . Then  $\psi_a$  has an inverse  $\chi_a$ , and the *S-transform* of  $a$  is defined as

$$S_a(z) = \frac{1+z}{z} \chi_a(z)$$

Observe that the power series  $zS_a(z)$  and  $R_a(z)$  are then inverse of each other (when the mean is nonzero). The relevance of these series to free probability is that, if  $a, b \in \mathcal{M}$  are free, then

$$R_{a+b} = R_a + R_b \quad \text{and} \quad S_{ab} = S_a S_b$$

see e.g. [VDN92].

**2.3. Calculus of  $R$ -diagonal elements.** We use the same notations as in the previous section.

**Definition 2.3.** A non-commutative random variable  $x$  is called  *$R$ -diagonal*, if  $x$  has polar decomposition  $x = uh$ , where  $u$  is a Haar unitary free from the radial part  $h = |x|$ .

Recall that a unitary  $u \in \mathcal{M}$  is called a Haar unitary if  $\tau(u^n) = 0$  for all integers  $n \neq 0$ . One can check that the product of an arbitrary element  $y$  with a free Haar unitary is an  $R$ -diagonal element. According to [HL99], any  $R$ -diagonal element with polar decomposition  $x = uh$  has the same distribution as a product  $a\tilde{h}$ , where  $\tilde{h}$  has a symmetric distribution, and its absolute value is distributed as  $h$ , whereas  $a$  is a self-adjoint unitary, free from  $\tilde{h}$ , and of zero trace. Indeed, one can assume  $\tilde{h} = a'h$ , where  $a'$  is a symmetry commuting with  $h$  and  $aa'$  is a Haar unitary free from  $h$ . Let  $a, b$  be two free  $R$ -diagonal elements, then one has equality in \*-distribution of the pairs  $(a, b)$  and  $(ua, ub)$  where  $u$  is a Haar unitary free with  $\{a, b\}$ , therefore  $a + b$  has the same \*-distribution as  $u(a + b)$  which is  $R$ -diagonal, and thus the sum of two free  $R$ -diagonal elements is again  $R$ -diagonal. Let  $f_x(z^2) = R_{\tilde{h}}(z)$  be the cumulant series of  $\tilde{h}$ , which determines the \*-distribution of  $x$ , then the power series  $z(1+z)S_{x^*x}(z)$  and  $f_x(z)$  are inverse of each other. Furthermore if  $a, b$  are two free  $R$ -diagonal elements, then one has

$$(2.1) \quad f_{a+b} = f_a + f_b.$$

See [NS97], [NS98] and [HL99].

**2.4. Brown measure of  $R$ -diagonal elements.** In [HL99] the Brown measure of an  $R$ -diagonal element is determined as follows.

**Theorem 2.4** ([HL99, Thm. 4.4, Prop. 4.6]). *Let  $u, h$  be \*-free random variables in  $(\mathcal{M}, \tau)$ , with  $u$  a Haar unitary and  $h$  positive s.t. the distribution  $\mu_h$  of  $h$  is not a Dirac measure. Then the Brown measure  $\mu_{uh}$  of  $uh$  has the following properties.*

- (i)  $\mu_{uh}$  is rotation invariant and its support is the annulus with inner radius  $\|h^{-1}\|_2^{-1}$  and outer radius  $\|h\|_2$ .
- (ii) The  $S$ -transform  $S_{\mu_{h^2}}$  of  $h^2$  has an analytic continuation to a neighbourhood of  $]\mu_h(\{0\}) - 1, 0]$  and its derivative  $S'_{\mu_{h^2}}$  is strictly negative on this interval and its range is  $S_{\mu_{h^2}}(]\mu_h(\{0\}) - 1, 0]) = [\|h\|_2^{-2}, \|h^{-1}\|_2^2]$ .
- (iii)  $\mu_{uh}(\{0\}) = \mu_h(\{0\})$  and for  $t \in ]\mu_h(\{0\}), 1]$

$$\mu_{uh} \left( B \left( 0, \frac{1}{\sqrt{S_{\mu_{h^2}}(t-1)}} \right) \right) = t$$

- (iv)  $\mu_{uh}$  is the only rotation symmetric probability measure satisfying (iii).
- (v) If  $h$  is invertible then  $\sigma(uh) = \text{supp } \mu_{uh}$ , i.e., the annulus discussed above.
- (vi) If  $h$  is not invertible then  $\sigma(uh) = B(0, \|h\|_2)$ .

The proof involves a formula for the spectral radius of products of free elements.

**Proposition 2.5** ([HL99, Prop. 4.1]). *Let  $a, b$  be \*-free centered elements in  $\mathcal{M}$ . Then the spectral radius of  $ab$  is*

$$\rho(ab) = \|a\|_2 \|b\|_2$$

In particular, an  $R$ -diagonal element  $a = uh$  can be written as  $u_1 u_2 h$ , with free Haar unitaries  $u_1, u_2$  and therefore its spectral radius is  $\rho(a) = \|u_1\|_2 \|u_2 h\|_2 = \|a\|_2$ .

### 3. ADDING AN $R$ -DIAGONAL ELEMENT

In this section we give a general approach to computing the Brown measure of the sum of a random variable with an arbitrary distribution and a free  $R$ -diagonal element. So we let  $a$  be an arbitrary element,  $h$  be self-adjoint and  $u$  a Haar unitary, with  $\{a, u, h\}$  forming a free family.

**3.1. The spectrum of  $a + uh$ .** The spectrum of  $a + uh$  is determined as follows. For  $\lambda \notin \sigma(a)$ ,  $\lambda - a - uh$  is invertible if and only if  $1 - uh(\lambda - a)^{-1}$  is invertible. If  $h$  is not invertible, then by the result of Haagerup and Larsen on  $R$ -diagonal elements, the latter is the case if and only if

$$(3.1) \quad \|h(\lambda - a)^{-1}\|_2 = \|h\|_2 \|(\lambda - a)^{-1}\|_2 < 1;$$

if  $h$  is invertible, we get the additional possibility that  $1 < \|h^{-1}\|_2 \|\lambda - a\|_2$ . In this case we can look at  $(uh)^{-1}(\lambda - a) - 1$ .

The case where  $\lambda \in \sigma(a)$  must be considered individually. Complications arise for such  $\lambda$ , for which  $\lambda \in \sigma(a)$ , but  $\|(\lambda - a)^{-1}\|_2 < \infty$ . Otherwise condition (3.1) will be satisfied when approaching  $\lambda$  from the outside of  $\sigma(a)$ , so that  $\lambda$  lies in the closure of the spectrum of  $a + uh$ , hence in the spectrum.

**3.2. The Brown measure of  $a + uh$ .** We can assume that  $u = u_1^* u_2$  with  $u_1$  and  $u_2$  Haar unitaries, where  $\{u_1, u_2, a, h\}$  is a free family, to get

$$\begin{aligned} \log \Delta(\lambda - a - uh) &= \tau(\log |u_1^*(u_1(\lambda - a) - u_2 h)|) \\ &= \tau(\log |u_1(\lambda - a) - u_2 h|) \\ &= \int \log |z| d\mu_{u_1(\lambda-a)-u_2h}(z) \end{aligned}$$

and this is the Fuglede–Kadison determinant of  $x_\lambda = u_1(\lambda - a) - u_2 h$ , which is an  $R$ -diagonal element whose  $*$ -distribution can be computed according to (2.1), i.e.  $f_x = f_{u_1|\lambda-a|} + f_{u_2 h}$ . This in turn will yield the  $S$ -transform of  $x_\lambda^* x_\lambda$ , and then by Theorem 2.4, we can compute  $\log \Delta(\lambda - a - uh)$ .

From the discussion in section 2.3 we have the relation

$$(3.2) \quad f_{x_\lambda}^{(-1)}(\zeta) = \frac{1}{\zeta} \left( 1 + R_{x_\lambda^* x_\lambda} \left( \frac{1}{\zeta} \right) \right)$$

In order to be more specific, let us assume that  $a$  is self-adjoint, then the computation of the distribution of  $(\lambda - a)^*(\lambda - a)$  is conveniently accomplished by using the Cauchy transform of  $a$ , namely factoring  $\zeta - |\lambda - x|^2 = (x - x_+)(x - x_-)$  with

$$(3.3) \quad x_\pm = \frac{1}{2} \left( \lambda + \bar{\lambda} \pm \sqrt{(\lambda - \bar{\lambda})^2 + 4\zeta} \right) = \operatorname{Re} \lambda \pm i\sqrt{(\operatorname{Im} \lambda)^2 - \zeta}$$

and expanding into partial fractions

$$\frac{1}{\zeta - |\lambda - x|^2} = \frac{1}{x_+ - x_-} \left( \frac{1}{x_+ - x} - \frac{1}{x_- - x} \right)$$

we get

$$(3.4) \quad G_{|\lambda-a|^2}(\zeta) = \int \frac{d\mu_a(x)}{\zeta - |\lambda - x|^2} = \frac{G_a(x_+) - G_a(x_-)}{x_+ - x_-}.$$

Using the same technique one can compute the 2-norm of the inverse of  $\lambda - a$ . Assuming again that  $a$  is self-adjoint we have that

$$\begin{aligned}
 \|\lambda - a\|_2^{-1} &= \int \frac{d\mu_a(x)}{|\lambda - x|^2} \\
 &= \int \frac{d\mu_a(x)}{(\lambda - x)(\bar{\lambda} - x)} \\
 (3.5) \quad &= \frac{1}{\lambda - \bar{\lambda}} \int \left( \frac{1}{\bar{\lambda} - x} - \frac{1}{\lambda - x} \right) d\mu_a(x) \\
 &= -\frac{G_a(\lambda) - G_a(\bar{\lambda})}{\lambda - \bar{\lambda}}
 \end{aligned}$$

Let us consider the simplest non-trivial random variable, namely  $a = u_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  having 2-point spectrum, so that  $|\lambda - u_2|^2$  has a Bernoulli distribution. The  $R$ -transform of  $|\lambda - u_2|^2 = 1 + |\lambda|^2 + (\lambda + \bar{\lambda})u_2$  is easily computed to be

$$R_{x_\lambda^* x_\lambda}(z) = (1 + |\lambda|^2) z + \frac{1}{2} \left( \sqrt{1 + 4(\lambda + \bar{\lambda})^2 z^2} - 1 \right)$$

and inverting it according to (3.2) leads to an equation of fourth degree, which apparently is unsuitable for further computations. So even this simple case seems to be untractable by this method. In fact, so far we have no concrete example where the general method above can be carried to the end. We shall develop other methods, in the next two sections, in order to treat the cases where the  $R$ -diagonal element is a Haar unitary, and a circular element.

#### 4. HAAR UNITARY CASE

Now  $a$  is an element with an arbitrary distribution, free with a Haar unitary  $u$ .

**4.1. The spectrum.** The spectrum of  $a + u$  is determined as follows: one has  $\lambda \in \sigma(a + u)$  if and only if  $1 \in \sigma(u^*(\lambda - a))$  and since the latter is  $R$ -diagonal, we infer from Theorem 2.4 that a necessary and sufficient condition is

$$(4.1) \quad \|\lambda - a\|_2^{-1} \leq 1 \leq \|\lambda - a\|_2$$

if  $\lambda \notin \sigma(a)$ ; otherwise the condition is simply  $1 \leq \|\lambda - a\|_2$ .

**4.2. First approach to the Fuglede-Kadison determinant.** We get the following formula for the Fuglede-Kadison determinant

$$\begin{aligned}
 \log \Delta(\lambda - a - u) &= \tau(\log |\lambda - a - u|) \\
 (4.2) \quad &= \tau(\log |u^*(\lambda - a) - 1|) \\
 &= \int \log |z - 1| d\mu_{u^*(\lambda-a)}(z).
 \end{aligned}$$

Observe that  $u^*(\lambda - a)$  is an  $R$ -diagonal element, and we can evaluate the integral as follows. The Brown measure of an  $R$ -diagonal element  $uh$  is rotationally symmetric with radial distribution  $\nu(dr)$  and one has

$$\int \log |z - 1| d\mu_{uh}(z) = \int_{\|h^{-1}\|_2^{-1}}^{\|h\|_2} \int_0^{2\pi} \log |re^{i\theta} - 1| d\theta \nu(dr)$$

where the inner integral reduces to

$$\frac{1}{2\pi} \int_0^{2\pi} \log |re^{i\theta} - 1| d\theta = \begin{cases} 0 & r < 1, \\ \log r & r \geq 1. \end{cases}$$

Introduce the radial distribution function

$$F_{uh}(r) = \mu_{uh}(B(0, r)) = 2\pi \int_{\|h^{-1}\|_2^{-1}}^r \nu(d\rho)$$

which according to Theorem 2.4 is related to the moment generating function  $\psi_{h^2}$  by

$$\psi_{h^2} \left( \frac{F_{uh}(r) - 1}{F_{uh}(r) r^2} \right) = F_{uh}(r) - 1$$

(for  $\|h^{-1}\|_2^{-1} \leq r \leq \|h\|_2$ ), and by partial integration (note that  $F(\|h\|_2) = 1$ )

$$\begin{aligned} \tau(\log |uh - 1|) &= \int_{\max(1, \|h^{-1}\|_2^{-1})}^{\|h\|_2} 2\pi \log(r) \nu(dr) \\ &= \log r F_{uh}(r) \Big|_{\max(1, \|h^{-1}\|_2^{-1})}^{\|h\|_2} - \int_{\max(1, \|h^{-1}\|_2^{-1})}^{\|h\|_2} \frac{F_{uh}(\rho)}{\rho} d\rho \\ &= \int_{\max(1, \|h^{-1}\|_2^{-1})}^{\|h\|_2} \frac{1 - F_{uh}(\rho)}{\rho} d\rho \end{aligned}$$

**Example 4.1** ( $2 \times 2$  matrix). Let  $a$  have the  $*$ -distribution of a  $2 \times 2$  matrix, and consider  $a + u$ ,  $u$  a Haar unitary. Let  $\mu_{\pm} = \mu_{\pm}(\lambda)$  be the eigenvalues of  $|\lambda - a|^2$  and let

$$G_{|\lambda-a|^2}(\zeta) = \frac{1}{2} \left( \frac{1}{\zeta - \mu_+} + \frac{1}{\zeta - \mu_-} \right)$$

be its Cauchy transform. Then

$$\psi(z) = \frac{1}{z} G \left( \frac{1}{z} \right) - 1 = \frac{1}{2} \left( \frac{1}{1 - \mu_+ z} + \frac{1}{1 - \mu_- z} \right) - 1$$

and we get  $F(r)$  by solving the equation  $\psi(\frac{t-1}{tr^2}) = t - 1$  for  $t$ :

$$\left( \frac{1}{1 - \mu_+ \frac{t-1}{tr^2}} + \frac{1}{1 - \mu_- \frac{t-1}{tr^2}} \right) = 2t$$

The obvious solution  $t = 1$  is not interesting for us, and dividing it out leads to the other solution

$$F(r) = \frac{2\mu_+ \mu_- - r^2(\mu_+ \mu_-)}{2(r^2 - \mu_+)(r^2 - \mu_-)} = \frac{\det |\lambda - a|^2 - r^2 \tau(|\lambda - a|^2)}{\det(r^2 - |\lambda - a|^2)}$$

The logarithm of the Fuglede–Kadison determinant is, for  $\lambda \in \sigma(a + u)$ ,

$$\begin{aligned}
\tau(\log |\lambda - a - u|) &= \int_1^{\|\lambda-a\|_2} \frac{1 - F(r)}{r} dr \\
&= \int_1^{\|\lambda-a\|_2} \frac{1}{2} \left( \frac{r}{r^2 - \mu_+} + \frac{r}{r^2 - \mu_-} \right) dr \\
&= \frac{1}{4} (\log |r^2 - \mu_+| + \log |r^2 - \mu_-|) \Big|_1^{\|\lambda-a\|_2} \\
&= \frac{1}{4} (\log |\|\lambda-a\|_2^4 - \det |\lambda-a|^2| - \log |1 - 2\|\lambda-a\|_2^2 + \det |\lambda-a|^2|) \\
&= \frac{1}{2} \log \left| \frac{\mu_+ - \mu_-}{2} \right| - \frac{1}{4} (\log |1 - \mu_+| + \log |1 - \mu_-|)
\end{aligned}$$

It is now convenient to use the representation of the Laplacian in terms of  $\partial_\lambda = \frac{1}{2} \left( \frac{\partial}{\partial \operatorname{Re} \lambda} - i \frac{\partial}{\partial \operatorname{Im} \lambda} \right)$  and its adjoint, namely

$$\nabla^2 = \frac{\partial^2}{\partial(\operatorname{Re} \lambda)^2} + \frac{\partial^2}{\partial(\operatorname{Im} \lambda)^2} = 4\partial_{\bar{\lambda}}\partial_\lambda.$$

Then we have the formulae

$$\begin{aligned}
\partial_\lambda \|\lambda - a\|_2^2 &= \partial_\lambda \tau((\lambda - a)^*(\lambda - a)) \\
&= \tau(\bar{\lambda} - a^*) \\
\partial_\lambda \det(\lambda - a) &= \partial_\lambda ((\lambda - \lambda_1(a))(\lambda - \lambda_2(a))) \\
&= 2\lambda - \lambda_1(a) - \lambda_2(a) \\
&= 2\tau(\lambda - a)
\end{aligned}$$

and the density of the Brown measure of  $a + u$  is

(4.3)

$$\begin{aligned}
p_{a+u}(\lambda) &= \frac{2}{\pi} \partial_{\bar{\lambda}} \partial_\lambda (\log |\|\lambda-a\|_2^4 - \det |\lambda-a|^2| - \log |1 - 2\|\lambda-a\|_2^2 + \det |\lambda-a|^2|) \\
&= \frac{2}{\pi} \partial_{\bar{\lambda}} \left( \frac{2\|\lambda-a\|_2^2 \tau(\bar{\lambda} - a^*) - 2\tau(\lambda - a) \det(\bar{\lambda} - a^*)}{\|\lambda-a\|_2^4 - \det |\lambda-a|^2} \right. \\
&\quad \left. - \frac{-2\tau(\bar{\lambda} - a^*) + 2\tau(\lambda - a) \det(\bar{\lambda} - a^*)}{1 - 2\|\lambda-a\|_2^2 + \det |\lambda-a|^2} \right) \\
&= \frac{4}{\pi} \left( \frac{\|\lambda-a\|_2^2 - |\tau(\lambda-a)|^2}{\|\lambda-a\|_2^4 - \det |\lambda-a|^2} - 2 \frac{|\|\lambda-a\|_2^2 \tau(\bar{\lambda} - a^*) - \det(\bar{\lambda} - a^*) \tau(\lambda - a)|^2}{(\|\lambda-a\|_2^4 - \det |\lambda-a|^2)^2} \right. \\
&\quad \left. - \frac{2|\tau(\lambda-a)|^2 - 1}{1 - 2\|\lambda-a\|_2^2 + \det |\lambda-a|^2} + 2 \frac{|\tau(\bar{\lambda} - a^*) - \tau(\lambda - a) \det(\bar{\lambda} - a^*)|^2}{(1 - 2\|\lambda-a\|_2^2 + \det |\lambda-a|^2)^2} \right)
\end{aligned}$$

and in terms of eigenvalues

$$\begin{aligned}
(4.4) \quad p_{a+u}(\lambda) &= \frac{2}{\pi} \partial_{\bar{\lambda}} \partial_{\lambda} \left( \frac{1}{2} \log \left| \frac{\mu_+ - \mu_-}{2} \right| - \frac{1}{4} (\log |1 - \mu_+| + \log |1 - \mu_-|) \right) \\
&= \frac{1}{\pi} \partial_{\bar{\lambda}} \left( \frac{1}{\mu_+ - \mu_-} \partial_{\lambda}(\mu_+ - \mu_-) + \frac{1}{2} \left( \frac{1}{1 - \mu_+} \partial_{\lambda} \mu_+ + \frac{1}{1 - \mu_-} \partial_{\lambda} \mu_- \right) \right) \\
&= \frac{1}{\pi} \left( \frac{\partial_{\bar{\lambda}} \partial_{\lambda}(\mu_+ - \mu_-)}{\mu_+ - \mu_-} - \left| \frac{\partial_{\lambda}(\mu_+ - \mu_-)}{\mu_+ - \mu_-} \right|^2 \right. \\
&\quad \left. + \frac{1}{2} \left( \frac{\partial_{\bar{\lambda}} \partial_{\lambda} \mu_+}{1 - \mu_+} + \frac{\partial_{\bar{\lambda}} \partial_{\lambda} \mu_-}{1 - \mu_-} + \left| \frac{\partial_{\lambda} \mu_+}{1 - \mu_+} \right|^2 + \left| \frac{\partial_{\lambda} \mu_-}{1 - \mu_-} \right|^2 \right) \right)
\end{aligned}$$

In particular, if  $a = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$  (Bernoulli distribution) one gets  $\mu_{\pm} = \{|\lambda - \alpha|^2, |\lambda - \beta|^2\}$  and consequently the density is

$$p_{a+u}(\lambda) = -\frac{|\beta - \alpha|^2}{\pi (|\lambda - \alpha|^2 - |\lambda - \beta|^2)^2} + \frac{1}{2\pi} \left( \frac{1}{(1 - |\lambda - \alpha|^2)^2} + \frac{1}{(1 - |\lambda - \beta|^2)^2} \right)$$

on the spectrum, which is determined by the inequalities

$$(4.5) \quad \frac{1}{\mu_+} + \frac{1}{\mu_-} \geq 2 \quad \mu_+ + \mu_- \geq 2$$

Specifying further  $\alpha = 1, \beta = -1$ , so that  $a$  is a symmetry, the spectrum is the region bounded by the lemniscate-like curve in the complex plane with the equation

$$|\lambda|^2 + 1 = |\lambda^2 - 1|^2$$

and we get the picture shown in figure 1.

This should be compared with the sample fig. 2 of eigenvalues of random  $2N \times 2N$  matrices of the form  $X = U_2 + U_{\infty}$ , where  $U_{\infty}$  is chosen with the Haar measure on  $U(2N)$ , and  $U_2 = V \Lambda V^*$ , with  $V$  a Haar distributed unitary independent of  $U_{\infty}$ , and  $\Lambda$  a fixed symmetry of trace zero.

As another example, specify to  $a = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$ . As we will see, the spectrum and Brown measure are radially symmetric. The eigenvalues of  $|\lambda - a|^2$  are

$$(4.6) \quad \mu_{\pm} = \frac{t^2 + 2|\lambda|^2 \pm t\sqrt{t^2 + 4|\lambda|^2}}{2}$$

and hence, substituting this into (4.5), we get

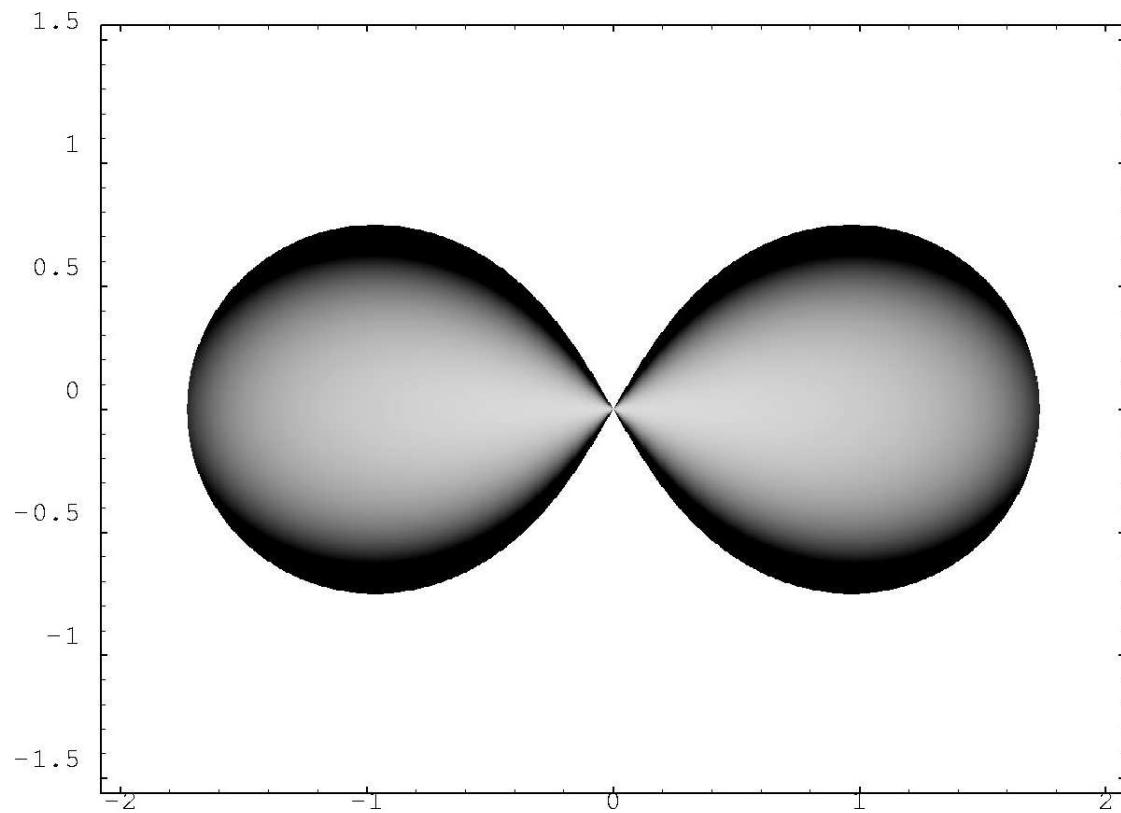
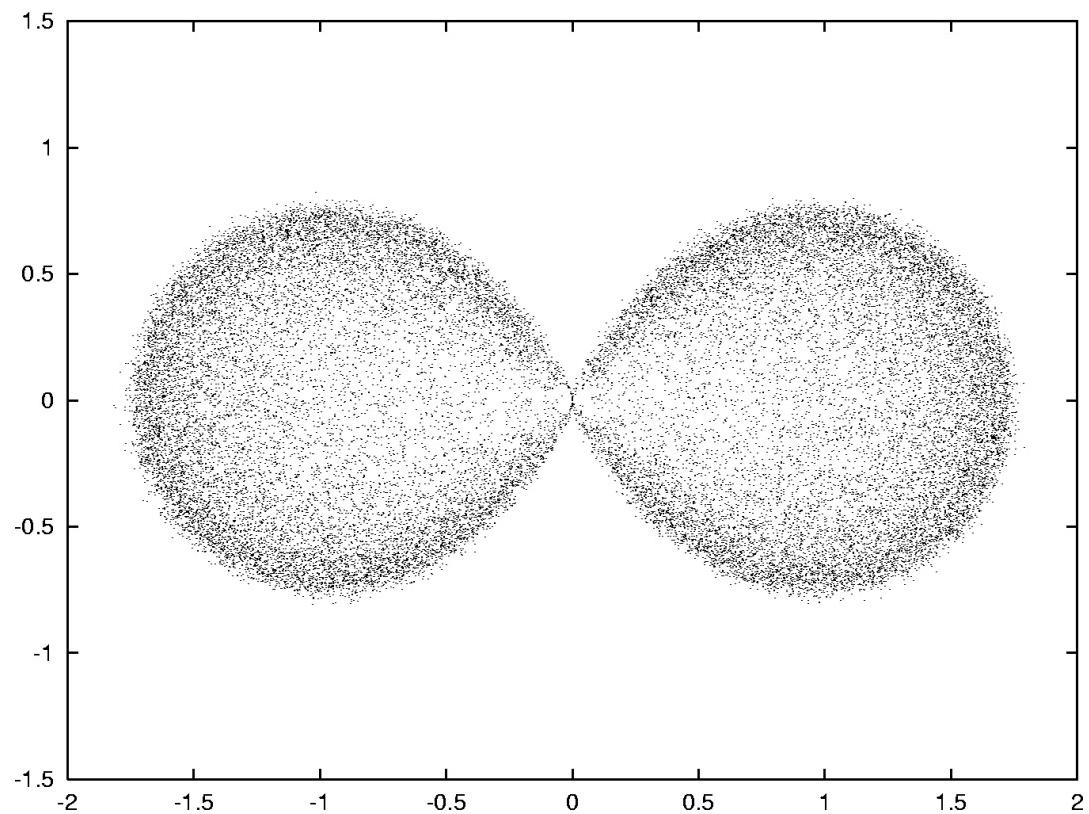
$$\sigma(a + u) = \left\{ \lambda : 1 - \frac{t^2}{2} \leq |\lambda|^2 \leq \sqrt{\frac{t^2}{2} + \frac{1}{4}} + \frac{1}{2} \right\}$$

which is a full disk for  $t \geq \sqrt{2}$  and an annulus otherwise. For the density we substitute the parameters

$$\|\lambda - a\|_2^2 = |\lambda|^2 + \frac{t^2}{2} \quad \det |\lambda - a|^2 = |\lambda|^4$$

into formula (4.3) we get the radially symmetric density function

$$p_{a+u}(\lambda) = \frac{4}{\pi} \left( \frac{2t^2}{(4|\lambda|^2 + t^2)^2} + \frac{(1 - |\lambda|^2)^2 - (1 - 2|\lambda|^2)t^2}{((1 - |\lambda|^2)^2 - t^2)^2} \right)$$

FIGURE 1. Density of  $\mu_{U_2 + U_\infty}$ FIGURE 2. 200 samples of eigenvalues of  $150 \times 150$  random matrices  $U_2 + U_\infty$

**4.3. An alternative expression for the Fuglede-Kadison determinant.** In order to treat more complicated examples, instead of the integral (4.2) it will be more convenient to use a more direct formula for the Kadison-Fuglede determinant, which we state as a lemma.

**Lemma 4.2** ([HL99, Proof of Theorem 4.4]). *Let  $uh$  be an  $R$ -diagonal element and define functions on  $\mathbf{R}_+ \setminus \{0\}$  by*

$$\begin{aligned} f(v) &= \tau((1 + vh^2)^{-1}) \\ g(v) &= \frac{1 - f(v)}{vf(v)} \end{aligned}$$

*Then  $g(v)$  is strictly decreasing with  $g([0, \infty[) = [\|h^{-1}\|_2^{-2}, \|h\|_2^2]$  and for every  $z \in [\|h^{-1}\|_2^{-2}, \|h\|_2^2[$  there is a unique  $v > 0$  such that  $z^2 = g(v)$ . With this  $v$  we have*

$$\log \Delta(uh - z) = \frac{1}{2} \int \log(1 + vt) d\mu_{h^2}(t) + \frac{1}{2} \log \frac{z^2}{1 + vz^2}$$

For our problem of computing  $\log \Delta(\lambda - a - u) = \log \Delta(u^*(\lambda - a) - 1)$  this translates as follows. Putting  $f(v, \lambda) = \tau((1 + v|a - \lambda|^2)^{-1})$  and denoting  $v(\lambda)$  the unique positive solution of the equation  $(1 + v)f(v, \lambda) = 1$ , then

$$\begin{aligned} \log \Delta(\lambda - a - u) &= \log \Delta(u^*(\lambda - a) - 1) \\ &= \frac{1}{2} \tau(\log(1 + v|a - \lambda|^2)) - \frac{1}{2} \log(1 + v). \end{aligned}$$

Note that this approach cannot be used in the general setting of section 3.2, as it does not tell how to evaluate the Kadison-Fuglede determinant at  $z = 0$ .

For the rest of this section we shall assume that  $a$  is normal with spectral measure  $\mu_a$ , so that we can write

$$(4.7) \quad f(v, \lambda) = \int \frac{d\mu_a(t)}{1 + v|\lambda - t|^2}$$

and again with  $(1 + v)f(v, \lambda) = 1$ ,

$$\log \Delta(a + u - \lambda) = \frac{1}{2} \int \log(1 + v|\lambda - t|^2) d\mu_a(t) - \frac{1}{2} \log(1 + v)$$

For the density of the Brown measure we obtain

$$\begin{aligned}
p(\lambda) &= \frac{2}{\pi} \partial_{\bar{\lambda}} \partial_{\lambda} \log \Delta(a + u - \lambda) \\
&= \frac{1}{\pi} \partial_{\bar{\lambda}} \left( \int \frac{|\lambda - t|^2 \partial_{\lambda} v + v(\bar{\lambda} - \bar{t})}{1 + v |\lambda - t|^2} d\mu(t) - \frac{1}{1 + v} \partial_{\lambda} v \right) \\
&= \frac{1}{\pi} \partial_{\bar{\lambda}} \left( \underbrace{\partial_{\lambda} v \left( \int \frac{|\lambda - t|^2}{1 + v |\lambda - t|^2} d\mu(t) - \frac{1}{1 + v} \right)}_{=0} + v \int \frac{\bar{\lambda} - \bar{t}}{1 + v |\lambda - t|^2} d\mu(t) \right) \\
&= \frac{1}{\pi} \partial_{\bar{\lambda}} \int \frac{1}{\lambda - t} \frac{v |\lambda - t|^2}{1 + v |\lambda - t|^2} d\mu(t) \\
&= \frac{1}{\pi} \partial_{\bar{\lambda}} \int \frac{1}{\lambda - t} \left( 1 - \frac{1}{1 + v |\lambda - t|^2} \right) d\mu(t) \\
&= \frac{1}{\pi} \int \frac{1}{\lambda - t} \frac{|\lambda - t|^2 \partial_{\bar{\lambda}} v + v(\lambda - t)}{(1 + v |\lambda - t|^2)^2} d\mu(t) \\
&= \frac{1}{\pi} \left( \partial_{\bar{\lambda}} v \int \frac{\bar{\lambda} - \bar{t}}{(1 + v |\lambda - t|^2)^2} d\mu(t) + \int \frac{v}{(1 + v |\lambda - t|^2)^2} d\mu(t) \right)
\end{aligned}$$

now by implicit differentiation

$$\begin{aligned}
1 &= (1 + v)f(v, \lambda) \\
0 &= \partial_{\bar{\lambda}} v f(v, \lambda) + (1 + v)(\partial_v f(v, \lambda) \partial_{\bar{\lambda}} v + \partial_{\bar{\lambda}} f(v, \lambda)) \\
\partial_{\bar{\lambda}} v &= -\frac{(1 + v) \partial_{\bar{\lambda}} f}{f + (1 + v) \partial_v f} \\
\partial_{\bar{\lambda}} f(v, \lambda) &= -\int \frac{v(\lambda - t)}{(1 + v |\lambda - t|^2)^2} d\mu(t) \\
\partial_v f(v, \lambda) &= -\int \frac{|\lambda - t|^2}{(1 + v |\lambda - t|^2)^2} d\mu(t)
\end{aligned}$$

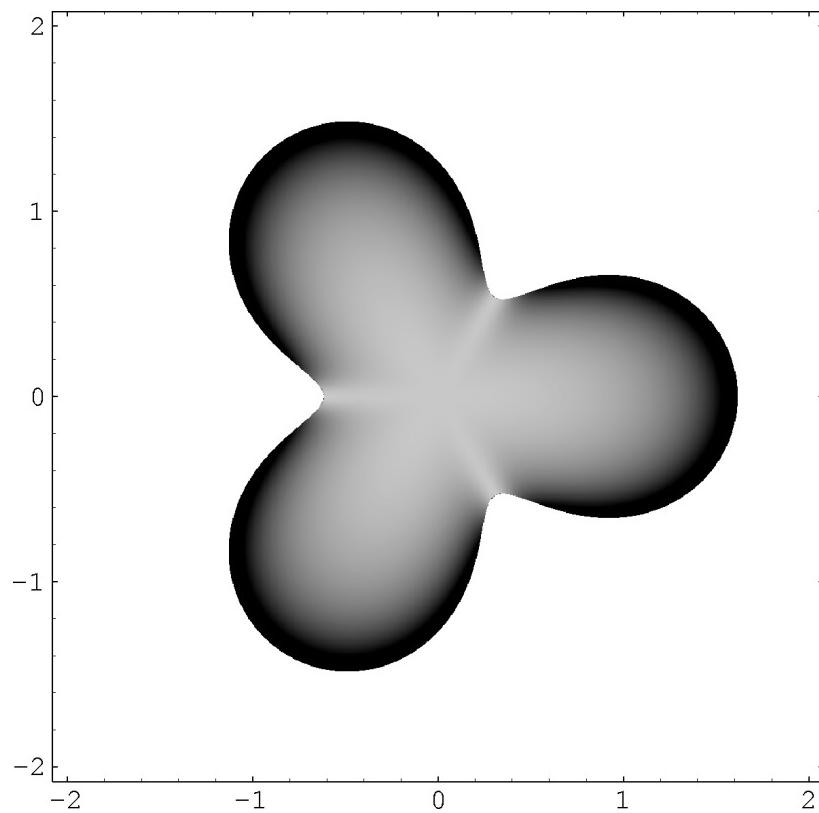
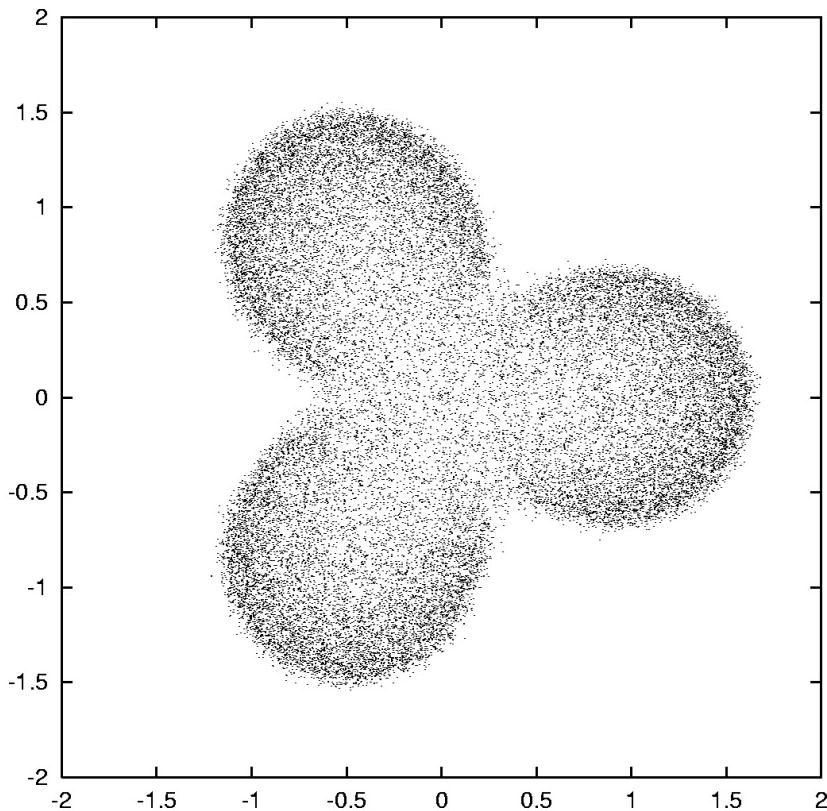
and thus

$$(4.8) \quad p(\lambda) = \frac{1}{\pi} \left( \frac{1 + v}{v(f(v, \lambda) + (1 + v) \partial_v f(v, \lambda))} |\partial_{\lambda} f(v, \lambda)|^2 + v f(v, \lambda) + v^2 \partial_v f(v, \lambda) \right)$$

We will apply this in three situations here. First consider a finite dimensional normal operator  $a$ , like e.g.  $a = u_n$ , the generator of the von Neumann algebra of  $\mathbf{Z}_n$ , then the integrals become finite sums and can be evaluated numerically. As an example see fig. 3, which should again be compared to the corresponding samples of spectra of random matrices in fig. 4. There  $U_3$  is a fixed  $150 \times 150$  permutation matrix with the same spectral distribution as  $u_3$  and  $U_\infty$  is again a  $150 \times 150$  standard unitary random matrix.

Secondly, assume that  $a$  is self-adjoint. Then we can factorize the denominator in the integral (4.7) as  $1 + v |\lambda - t|^2 = vt^2 - v(\lambda + \bar{\lambda})t + 1 + v|\lambda|^2 = v(t - z_0)(t - \bar{z}_0)$  where

$$z_0 = \operatorname{Re} \lambda + \frac{i}{v} \sqrt{v^2 (\operatorname{Im} \lambda)^2 + v}.$$

FIGURE 3. Density of  $\mu_{u_3+u_\infty}$ FIGURE 4. 200 samples of eigenvalues of  $150 \times 150$  random matrices  $U_3 + U_\infty$

From this we can express  $f(v, \lambda)$  and therefore  $p(\lambda)$  in terms of the Cauchy transform  $G(\zeta)$  of  $a$  as follows.

$$\begin{aligned} f(v, \lambda) &= \int \frac{d\mu(t)}{v(t - z_0)(t - \bar{z}_0)} \\ &= \frac{1}{v} \int \frac{1}{z_0 - \bar{z}_0} \left( \frac{1}{t - z_0} - \frac{1}{t - \bar{z}_0} \right) d\mu(t) \\ &= -\frac{\operatorname{Im} G(z_0)}{\sqrt{v^2(\operatorname{Im} \lambda)^2 + v}} \end{aligned}$$

As an example consider  $a = u_2 + v_2$ , where  $u_2$  and  $v_2$  are the generators of two free copies of  $\mathbf{Z}_2$ . Then  $a$  is self-adjoint and distributed according to the arcsine law (or Kesten measure) and has Cauchy transform  $G(\zeta) = \frac{1}{\zeta \sqrt{1 - \frac{1}{\zeta^2}}}$ . A picture of the density of the Brown measure of  $u_2 + v_2 + u$  is presented in fig. 5.

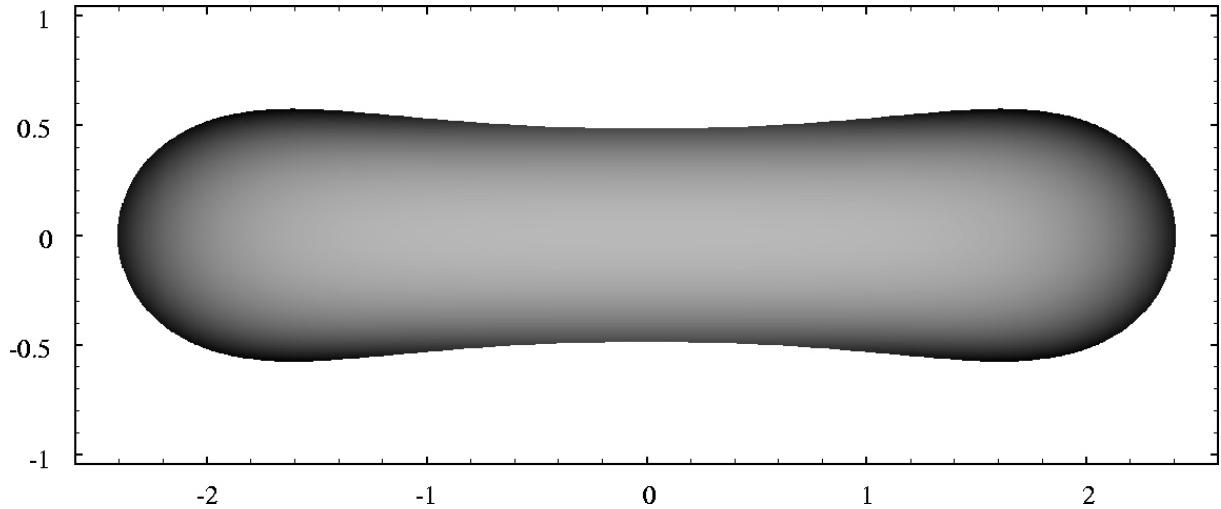


FIGURE 5. Density of  $\mu_{u_2+v_2+u_\infty}$

Finally, let us consider the free sum of an arbitrary unitary  $v$  and a Haar unitary  $u$ . Let  $d\mu(\theta)$  be the spectral measure of  $v$  on the unit circle. For the evaluation of the integral (4.7) we factorize the denominator again, this time writing

$$\begin{aligned} f(v, \lambda) &= \int \frac{d\mu(\theta)}{1 + v |\lambda - e^{i\theta}|^2} \\ &= \int \frac{d\mu(\theta)}{1 + v(|\lambda|^2 + 1) - v(\lambda e^{-i\theta} + \bar{\lambda} e^{i\theta})} \\ &= - \int \frac{e^{i\theta}}{v \bar{\lambda} e^{2i\theta} - (1 + v(|\lambda|^2 + 1)) e^{i\theta} + v \lambda} d\mu(\theta) \\ &= -\frac{1}{v \bar{\lambda}} \int \frac{e^{i\theta}}{(e^{i\theta} - z_+)(e^{i\theta} - z_-)} \end{aligned}$$

where

$$z_\pm = \frac{1}{2v\bar{\lambda}} \left( 1 + v(|\lambda|^2 + 1) \pm \sqrt{(1 + v(|\lambda|^2 + 1)^2)(1 + v(|\lambda|^2 - 1)^2)} \right).$$

Note that  $|z_+ z_-| = |\frac{\lambda}{\bar{\lambda}}|$  and  $|z_+| > |z_-|$ , and thus  $|z_+| > 1 > |z_-|$ .

$$\begin{aligned} f(v, \lambda) &= \frac{1}{v\bar{\lambda}} \int \frac{e^{i\theta}}{z_+ z_-} \left( \frac{1}{(z_+ - e^{i\theta})} - \frac{1}{(z_- - e^{i\theta})} \right) d\mu(\theta) \\ &= \frac{1}{v\bar{\lambda}} \int \frac{1}{z_+ z_-} \left( \frac{z_+}{(z_+ - e^{i\theta})} - \frac{z_-}{(z_- - e^{i\theta})} \right) d\mu(\theta) \\ &= \frac{z_+ G(z_+) - z_- G(z_-)}{v\bar{\lambda}(z_+ - z_-)} \\ &= \frac{z_+ G(z_+) - z_- G(z_-)}{\sqrt{(1 + v(|\lambda|^2 + 1)^2)(1 + v(|\lambda|^2 - 1)^2)}} \end{aligned}$$

For the determination of the spectrum (4.1) we need

$$\begin{aligned} \|(\lambda - v)^{-1}\|_2^2 &= \int \frac{d\mu(\theta)}{|\lambda - e^{i\theta}|^2} \\ &= \frac{1}{|\lambda|^2 - 1} \int \left( \frac{\lambda}{\lambda - e^{i\theta}} + \frac{\bar{\lambda}}{\bar{\lambda} - e^{-i\theta}} - 1 \right) d\mu(\theta) \\ &= \frac{\lambda G(\lambda) + \bar{\lambda} G(\bar{\lambda}) - 1}{|\lambda|^2 - 1} \end{aligned}$$

As an example let us consider for  $q \in [-1, 1]$  the unitary  $u_q$  with Poisson distribution, i.e. whose moments are  $\tau(u_q^n) = q^{|n|}$ . For  $q = 0$  this is the Haar distribution, while for  $q = 1$  it is the Dirac measure at 1. By Fourier transform, the density of the spectral measure is

$$d\mu_q(\theta) = \frac{1}{2\pi} \frac{1 - q^2}{|1 - qe^{i\theta}|^2}.$$

The Cauchy transform is

$$G_q(\zeta) = \begin{cases} \frac{1}{\zeta - q} & |\zeta| > 1 \\ \frac{1}{\zeta - q^{-1}} & |\zeta| < 1 \end{cases}$$

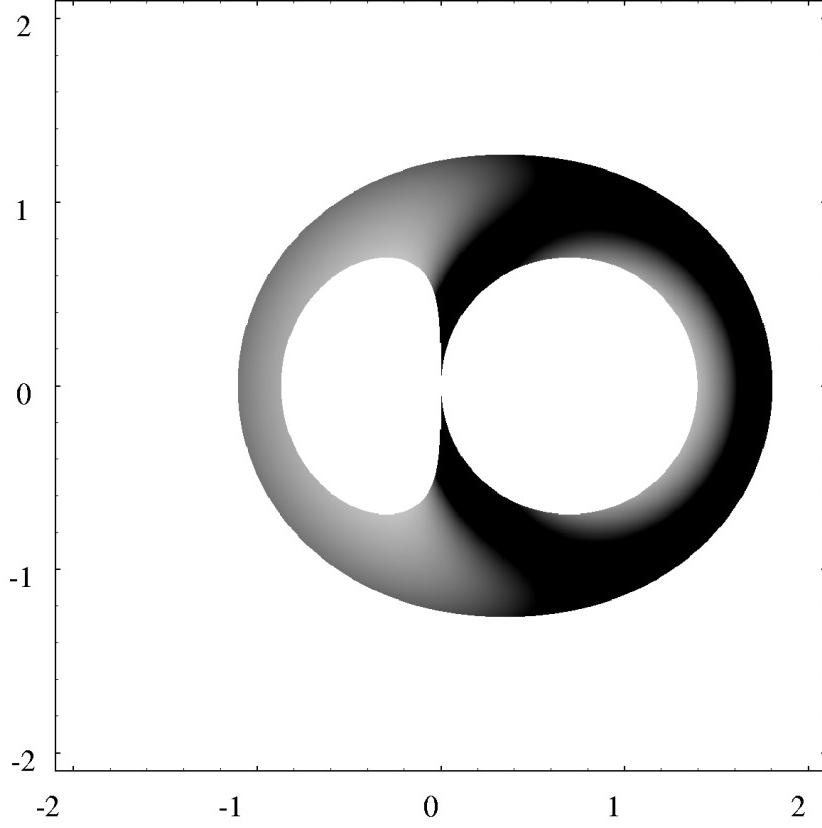
and from this we get the other relevant functions

$$\begin{aligned} \|(\lambda - u_q)^{-1}\|_2^2 &= \begin{cases} \frac{|\lambda|^2 - q^2}{(|\lambda|^2 - 1)|\lambda - q|^2} & |\lambda| > 1 \\ \frac{q^{-2} - |\lambda|^2}{(1 - |\lambda|^2)|\lambda - q^{-1}|^2} & |\lambda| < 1 \end{cases} \\ f(v, \lambda) &= \frac{qz_+ - q^{-1}z_-}{(z_+ - q)(z_- - q^{-1})} \frac{1}{\sqrt{(1 + v(|\lambda|^2 + 1)^2)(1 + v(|\lambda|^2 - 1)^2)}} \end{aligned}$$

Substituting this into (4.8), we get pictures like fig. 6, where  $q = 0.7$ .

## 5. ADDING A CIRCULAR ELEMENT

A standard circular element has the  $*$ -distribution of  $C = S_1 + iS_2$  where  $S_1, S_2$  are free standard semi-circular elements, i.e., self-adjoints whose distribution is the semi-circle law  $\frac{1}{2\pi} \sqrt{4 - x^2} dx$  on  $[-2, +2]$ . Its polar decomposition is  $C = uh$  with  $u$  a Haar unitary free with  $h$  (hence  $C$  is  $R$ -diagonal), and  $h$  has the quarter circular distribution  $\frac{1}{\sqrt{2\pi}} \sqrt{8 - x^2} dx$  on  $[0, \sqrt{8}]$ . The symmetrized  $\tilde{h}$  in Haagerup-Larsen's decomposition  $C = a\tilde{h}$  has a semi-circular distribution of variance 2. In this section we consider the Brown measure of  $X_t = X_0 + C_t$ , where  $X_0$  has arbitrary  $*$ -distribution, it is free with  $C_t$  and  $C_t$  is a circular element of variance  $t$ , i.e.  $C_t \cong \sqrt{\frac{t}{2}} C$  where  $C$  is a standard circular element. It will be convenient to assume that

FIGURE 6. Density of  $\mu_{u_q+u_\infty}$  at  $q = 0.7$ 

the  $C_t$  form a circular process, i.e., for each  $s < t$ ,  $C_t - C_s$  is  $*$ -free with  $C_s$ . We shall use a heat equation like approach, by differentiating in  $t$ . One has

$$\log \Delta(\lambda - X_t) = \frac{1}{2} \log \Delta(|\lambda - X_t|^2) = \frac{1}{2} \lim_{\varepsilon \rightarrow 0} \log \Delta(|\lambda - X_t|^2 + \varepsilon^2).$$

Let us denote  $H_{t,\varepsilon} = |\lambda - X_t|^2 + \varepsilon^2$  and compute the derivative  $\frac{\partial}{\partial t} \log \Delta(H_{t,\varepsilon})$ . To this end let  $dt$  be small,  $dC_t = C_{t+dt} - C_t$  (so that  $\tau(dC_t^* dC_t) = dt$ ). Then

$$\begin{aligned} H_{t+dt,\varepsilon} &= |\lambda - X_{t+dt}|^2 + \varepsilon^2 \\ &= |\lambda - X_t - dC_t|^2 + \varepsilon^2 \\ &= |\lambda - X_t|^2 - (\lambda - X_t)^* dC_t - dC_t^* (\lambda - X_t) + |dC_t|^2 + \varepsilon^2 \\ &= H_{t,\varepsilon} - (\lambda - X_t)^* dC_t - dC_t^* (\lambda - X_t) + dC_t^* dC_t \\ &= H_{t,\varepsilon} [1 - H_{t,\varepsilon}^{-1}((\lambda - X_t)^* dC_t + dC_t^* (\lambda - X_t) - dC_t^* dC_t)] \end{aligned}$$

and hence

$$\begin{aligned} \log \Delta(H_{t+dt,\varepsilon}) &= \log \Delta(H_{t,\varepsilon}) + \log \Delta(1 - H_{t,\varepsilon}^{-1}((\lambda - X_t)^* dC_t + dC_t^* (\lambda - X_t) - dC_t^* dC_t)) \\ &= \log \Delta(H_{t,\varepsilon}) + \tau(\log |1 - H_{t,\varepsilon}^{-1}((\lambda - X_t)^* dC_t + dC_t^* (\lambda - X_t) - dC_t^* dC_t)|). \end{aligned}$$

Now observe that

$$\begin{aligned}
\tau(\log |1 + a\sqrt{dt} + b dt|) &= \frac{1}{2}\tau(\log |1 + a\sqrt{dt} + b dt|^2) \\
&= \frac{1}{2}\tau\left(\log(1 + (a + a^*)\sqrt{dt} + (b + b^*)dt + a^*a dt + \mathcal{O}((dt)^{3/2}))\right) \\
&= \frac{1}{2}\tau\left((a + a^*)\sqrt{dt} + (b + b^* + a^*a)dt - \frac{1}{2}(a + a^*)^2dt\right) + \mathcal{O}((dt)^{3/2}) \\
&= \frac{1}{2}\tau\left((a + a^*)\sqrt{dt} + (b + b^* - \frac{a^2 + a^{*2}}{2})dt\right) + \mathcal{O}((dt)^{3/2})
\end{aligned}$$

In our situation we have  $a = -H_{t,\varepsilon}^{-1}\left((\lambda - X_t)^*\frac{dC_t}{\sqrt{dt}} + \frac{dC_t}{\sqrt{dt}}*(\lambda - X_t)\right)$  and  $b = H_{t,\varepsilon}^{-1}\frac{dC_t^* dC_t}{dt}$ , so that  $\tau(a) = 0$  and  $\tau(b) = \tau(H_{t,\varepsilon}^{-1})$  by freeness of  $dC_t$  and  $\{H_{t,\varepsilon}, \lambda - X_t\}$ . Further we have

$$\begin{aligned}
\tau(a^2) &= \tau\left(\left(H_{t,\varepsilon}^{-1}(\lambda - X_t)^*\frac{dC_t}{\sqrt{dt}}\right)^2 + \left(H_{t,\varepsilon}^{-1}\frac{dC_t^*}{\sqrt{dt}}(\lambda - X_t)\right)^2\right. \\
&\quad \left.+ 2H_{t,\varepsilon}^{-1}(\lambda - X_t)^*\frac{dC_t}{\sqrt{dt}}H_{t,\varepsilon}^{-1}\frac{dC_t^*}{\sqrt{dt}}(\lambda - X_t)\right)
\end{aligned}$$

and using the formula  $\tau(a_1 b_1 a_2 b_2) = \tau(a_1)\tau(a_2)\tau(b_1 b_2)$  if  $\{a_1, a_2\}$  is free from  $\{b_1, b_2\}$  and  $\tau(b_1) = \tau(b_2) = 0$ , we see that only the last term is nonzero and equal to

$$\begin{aligned}
&= 2\tau\left((\lambda - X_t)H_{t,\varepsilon}^{-1}(\lambda - X_t)^*\frac{dC_t}{\sqrt{dt}}H_{t,\varepsilon}^{-1}\frac{dC_t^*}{\sqrt{dt}}\right) \\
&= 2\tau((\lambda - X_t)H_{t,\varepsilon}^{-1}(\lambda - X_t)^*)\tau(H_{t,\varepsilon}^{-1}) \\
&= 2\tau(H_{t,\varepsilon}^{-1}(H_{t,\varepsilon} - \varepsilon^2)\tau(H_{t,\varepsilon}^{-1}) \\
&= 2\tau(H_{t,\varepsilon}^{-1}) + 2\varepsilon^2\tau(H_{t,\varepsilon}^{-1})^2
\end{aligned}$$

so that

$$\begin{aligned}
\frac{\log \Delta(H_{t+dt,\varepsilon}) - \log \Delta(H_{t,\varepsilon})}{dt} &= \frac{1}{2}(2\tau(H_{t,\varepsilon}^{-1}) - 2\tau(H_{t,\varepsilon}^{-1}) + 2\varepsilon^2\tau(H_{t,\varepsilon}^{-1})^2 + \mathcal{O}((dt)^{1/2})) \\
&= \varepsilon^2\tau(H_{t,\varepsilon}^{-1})^2 + \mathcal{O}((dt)^{1/2})
\end{aligned}$$

hence

$$\frac{\partial}{\partial t} \log \Delta(H_{t,\varepsilon}) = \varepsilon^2\tau(H_{t,\varepsilon}^{-1})^2$$

and

$$\log \Delta(H_{t,\varepsilon}^{-1}) = \log \Delta(H_{0,\varepsilon}^{-1}) + \int_0^t \varepsilon^2 \tau(H_{s,\varepsilon}^{-1})^2 ds.$$

Let  $a_{\lambda,s}$  be a self-adjoint element with symmetric distribution, whose absolute value is distributed as  $|\lambda - X_s|$ . Now note that by the Stieltjes inversion formula

$$\begin{aligned}
\varepsilon \tau(H_{s,\varepsilon}^{-1}) &= \tau\left(\varepsilon(|\lambda - X_s|^2 + \varepsilon^2)^{-1}\right) \\
&= -\tau\left(\text{Im}[(i\varepsilon - a_{\lambda,s})^{-1}]\right) \\
&\xrightarrow{\varepsilon \rightarrow 0} \pi \left.\frac{d\mu_{a_{\lambda,s}}(x)}{dx}\right|_{x=0}
\end{aligned}$$

i.e., the density at 0 of the distribution of  $a_{\lambda,s}$ . Now we need the following

**Lemma 5.1.** *Let  $a$  be a selfadjoint symmetrically distributed element, free with  $S$  and  $C$ , where  $S$  and  $C$  are a semi-circular and a circular element of same variance respectively, then  $|a + S|$  and  $||a| + C|$  have the same distribution.*

*Proof.* Let  $b$  be a symmetry free with  $\{a, S, C\}$ , then by [HL99, Prop. 4.2]  $ba$  and  $bS$  are  $*$ -free, thus  $|a + S| = |ba + bS|$  is distributed as  $|ba + C|$ . Now using the fact that multiplying with a free Haar unitary  $u$  does not change the  $*$ -distribution of  $C$ , we can replace the latter according to  $C \cong u^*C$ , and get the following equalities of  $*$ -distributions

$$|ba + C| \cong |ba + u^*C| \cong |uba + C| \cong |u|a| + C| \cong ||a| + C|$$

□

Using the lemma we get

$$|\lambda - X_s| = |\lambda - X_0 - C_s| \cong |a_\lambda + S_s|$$

where  $a_\lambda$  is the symmetrization of  $|\lambda - X_0|$ , free with the semicircular  $S_s$  and therefore

$$|a_{\lambda,s}| \cong |a_\lambda + S_s|$$

It follows from Corollary 3 of [Bia97, p. 711] that the distribution of  $a_{\lambda,s}$  has a density at 0 which is  $p_s(0) = \frac{v(s)}{\pi s}$ , with

$$(5.1) \quad v(s) = \inf \left\{ v \geq 0 : \int \frac{d\mu_{|\lambda-X_0|}(x)}{x^2 + v^2} \leq \frac{1}{s} \right\}$$

If  $\lambda \notin \sigma(X_0)$ , then by e.g. [Bia97]

$$\begin{aligned} \varepsilon\tau(H_{s,\varepsilon}^{-1}) &\leq \sup_{x \in \mathbf{R}} \frac{d\mu_{a_{\lambda,s}}(x)}{dx} \\ &\leq \frac{1}{\pi\sqrt{s}} \end{aligned}$$

furthermore for  $s$  small enough,  $\lambda \notin \sigma(X_s)$  and  $\tau(|\lambda - X_s\lambda|^{-2})$  is bounded above, hence  $\varepsilon\tau(H_{s,\varepsilon}^{-1})$  also, therefore we can apply the dominated convergence theorem and we get

$$\begin{aligned} (5.2) \quad \log \Delta(\lambda - X_t) &= \frac{1}{2} \lim_{\varepsilon \rightarrow 0} \log \Delta(H_{0,\varepsilon}) + \frac{1}{2} \int_0^t \varepsilon^2 \tau(H_{s,\varepsilon}^{-1})^2 ds \\ &= \log \Delta(\lambda - X_0) + \frac{1}{2} \int_0^t \frac{v(s)^2}{s^2} ds \\ &= \log \Delta(\lambda - X_0) + \frac{1}{2} \int_{t_\lambda}^t \frac{v(s)^2}{s^2} ds \end{aligned}$$

where  $t_\lambda = \inf\{t : v(t) > 0\} = (\int \frac{d\mu_{|\lambda-X_0|}(x)}{x^2})^{-1}$ . So whenever  $\lambda \notin \sigma(X_0)$ , the density of the Brown measure is

$$\begin{aligned} p_{\lambda-X_t}(\lambda) &= \frac{1}{\pi} \partial_{\bar{\lambda}} \partial_\lambda \int_{t_\lambda}^t \frac{v(s)^2}{s^2} ds \\ &= \frac{1}{\pi} \partial_{\bar{\lambda}} \left( \int_{t_\lambda}^t \frac{\partial_\lambda v(s)^2}{s^2} ds - \frac{v(t_\lambda)^2}{t_\lambda^2} \partial_\lambda t_\lambda \right) \end{aligned}$$

and the second summand will be zero if  $v(t)$  is continuous at  $t_\lambda$ .

**Example 5.2** ( $2 \times 2$  matrix). Let  $X_0 = a$  be as in example 4.1, and consider  $X_t = a + C_t$ . Let again  $\mu_{\pm}$  be the eigenvalues of  $(\lambda - a)^*(\lambda - a)$ , then the relevant parameters are

$$\begin{aligned}\|\lambda - a\|_2^2 &= \frac{\mu_+ + \mu_-}{2} \\ \|\lambda - a\|_2^2 &= \frac{1}{2} \left( \frac{1}{\mu_+} + \frac{1}{\mu_-} \right) = \frac{\|\lambda - a\|_2^2}{\det |\lambda - a|^2} \\ t_{\lambda} &= \left( \int \frac{d\mu_{|\lambda-a|}(x)}{x^2} dx \right)^{-1} = \|\lambda - a\|_2^{-2} = \frac{\det |\lambda - a|^2}{\|\lambda - a\|_2^2}.\end{aligned}$$

The function  $v(s)^2$  is the solution of the quadratic equation

$$\begin{aligned}\frac{1}{s} &= \frac{1}{2} \left( \frac{1}{\mu_+ + v^2} + \frac{1}{\mu_- + v^2} \right) \\ &= \frac{1}{2} \frac{\mu_+ + \mu_- + 2v^2}{\mu_+ \mu_- + (\mu_+ + \mu_-)v^2 + v^4} \\ &= \frac{\|\lambda - a\|_2^2 + v^2}{\det |\lambda - a|^2 + 2\|\lambda - a\|_2^2 v^2 + v^4}\end{aligned}$$

which is explicitly

$$\begin{aligned}v(s)^2 &= \frac{1}{2} \left( s - 2\|\lambda - a\|_2^2 \pm \sqrt{(s - 2\|\lambda - a\|_2^2)^2 - 4(\det |\lambda - a|^2 - s\|\lambda - a\|_2^2)} \right) \\ &= \frac{1}{2} \left( s - 2\|\lambda - a\|_2^2 \pm \sqrt{s^2 + 4(\|\lambda - a\|_2^4 - \det |\lambda - a|^2)} \right)\end{aligned}$$

Now we have to choose the right branch of the square root. To this end, let us compute the spectrum of  $X_t$ : Assume  $\lambda \notin \sigma(a)$ , then  $\lambda \in \sigma(a + C_t)$  if and only if  $1 - C_t(\lambda - a)^{-1}$  is not invertible. Now  $C_t(\lambda - a)^{-1}$  is  $R$ -diagonal and not invertible, so by Theorem 2.4 (vi), 1 is in its spectrum if and only if its spectral radius is at least 1 and using Proposition 2.5 we get the inequality

$$1 \leq \rho(C_t(\lambda - a)^{-1}) = \|C_t\|_2 \|\lambda - a\|_2$$

in other words,

$$\det |\lambda - a|^2 \leq t \|\lambda - a\|_2^2$$

and hence for  $s < t$ ,  $\det |\lambda - a|^2 - s \|\lambda - a\|_2^2 < 0$ , only the “+” branch gives a nonnegative solution. Consequently

$$\begin{aligned}\log \Delta(\lambda - X_t) - \log \Delta(\lambda - X_0) &= \frac{1}{2} \int_{t_{\lambda}}^t \frac{1}{2s} - \frac{\|\lambda - a\|_2^2}{s^2} + \frac{\sqrt{s^2 + 4(\|\lambda - a\|_2^4 - \det |\lambda - a|^2)}}{2s^2} ds \\ &= \frac{1}{4} \log s + \frac{\|\lambda - a\|_2^2}{2s} + \frac{1}{4} \log \left( s + \sqrt{s^2 + 4(\|\lambda - a\|_2^4 - \det |\lambda - a|^2)} \right) \\ &\quad - \frac{1}{4s} \sqrt{s^2 + 4(\|\lambda - a\|_2^4 - \det |\lambda - a|^2)} \Big|_{s=t_{\lambda}}^t\end{aligned}$$

Now observe that

$$\sqrt{t_{\lambda}^2 + 4(\|\lambda - a\|_2^4 - \det |\lambda - a|^2)} = \frac{2\|\lambda - a\|_2^4 - \det |\lambda - a|^2}{\|\lambda - a\|_2^2}$$

and hence, denoting

$$(5.3) \quad R(\lambda) = 4(\|\lambda - a\|_2^4 - \det |\lambda - a|^2) = (\mu_+ - \mu_-)^2$$

we get

$$\begin{aligned} & \log \Delta(\lambda - X_t) - \log \Delta(\lambda - X_0) \\ &= \frac{1}{4} \log t + \frac{\|\lambda - a\|_2^2}{2t} + \frac{1}{4} \log \left( t + \sqrt{t^2 + R(\lambda)} \right) - \frac{1}{4t} \sqrt{t^2 + R(\lambda)} \\ &\quad - \frac{1}{4} \log \frac{\det |\lambda - a|^2}{\|\lambda - a\|_2^2} - \frac{\|\lambda - a\|_2^4}{2 \det |\lambda - a|^2} - \frac{1}{4} \log \frac{\det |\lambda - a|^2 + 2\|\lambda - a\|_2^4 - \det |\lambda - a|^2}{\|\lambda - a\|_2^2} \\ &\quad + \frac{\|\lambda - a\|_2^2}{4 \det |\lambda - a|^2} \frac{2\|\lambda - a\|_2^4 - \det |\lambda - a|^2}{\|\lambda - a\|_2^2} \\ &= \frac{1}{4} \log t + \frac{\|\lambda - a\|_2^2}{2t} + \frac{1}{4} \log \left( t + \sqrt{t^2 + R(\lambda)} \right) - \frac{1}{4t} \sqrt{t^2 + R(\lambda)} \\ &\quad - \frac{1}{4} \log \det |\lambda - a|^2 - \frac{1}{4} \log 2 - \frac{1}{4} \end{aligned}$$

and finally the density is (note that  $\partial_{\bar{\lambda}} \partial_{\lambda} \log \det |\lambda - a|^2 = 0$  and  $\partial_{\bar{\lambda}} \partial_{\lambda} \|\lambda - a\|_2^2 = 1$ )

$$\begin{aligned} p_{a+C_t}(\lambda) &= \frac{2}{\pi} \partial_{\bar{\lambda}} \partial_{\lambda} \log \Delta(\lambda - X_t) \\ &= \frac{1}{\pi t} + \frac{1}{2\pi} \partial_{\bar{\lambda}} \partial_{\lambda} \left( \log \left( t + \sqrt{t^2 + R(\lambda)} \right) - \frac{\sqrt{t^2 + R(\lambda)}}{t} \right) \\ &= \frac{1}{\pi t} + \frac{1}{2\pi} \partial_{\bar{\lambda}} \left( \frac{1}{t + \sqrt{t^2 + R(\lambda)}} - \frac{1}{t} \right) \frac{\partial_{\lambda} R(\lambda)}{2\sqrt{t^2 + R(\lambda)}} \\ &= \frac{1}{\pi t} + \frac{1}{4\pi} \partial_{\bar{\lambda}} \left( \frac{t - \sqrt{t^2 + R(\lambda)}}{tR(\lambda)} \partial_{\lambda} R(\lambda) \right) \\ &= \frac{1}{\pi t} + \frac{1}{4\pi t} \left( -\frac{|\partial_{\lambda} R(\lambda)|^2}{2R(\lambda)\sqrt{t^2 + R(\lambda)}} + \left( t - \sqrt{t^2 + R(\lambda)} \right) \frac{R(\lambda)\partial_{\bar{\lambda}} \partial_{\lambda} R(\lambda) - |\partial_{\lambda} R(\lambda)|^2}{R(\lambda)^2} \right) \end{aligned}$$

Again we can specify to  $a = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ , and we get the spectrum

$$\sigma(a + C_t) = \{\lambda : |\lambda^2 - 1|^2 \leq t(|\lambda|^2 + 1)\}$$

Note that for  $t = 1$  this is the same as  $\sigma(u_2 + u)$  from example 4.1. However this time the density is a function of the real part alone, namely substituting  $\mu_{\pm} = |\lambda \pm 1|^2$  into (5.3), we get  $R(\lambda) = 4(\lambda + \bar{\lambda})^2$  and consequently the density depends only on the real part

$$p_{a+C_t}(x + iy) = \frac{1}{\pi t} + \frac{1}{8\pi x^2} \left( \frac{t}{\sqrt{t^2 + 16x^2}} - 1 \right)$$

The situation for the nilpotent  $2 \times 2$  matrix  $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  is as follows. We have computed the eigenvalues of  $|\lambda - a|^2$  in (4.6), and thus

$$\sigma(a + C_t) = \{\lambda : 2|\lambda|^4 \leq t(1 + 2|\lambda|^2)\}$$

which is the disk with radius  $\sqrt{\sqrt{\frac{t^2}{4} + \frac{1}{2}} + \frac{t}{2}}$ . This is the same as  $\sigma(a + \sqrt{t}u)$ , but with the possible hole removed. Furthermore we get  $R(\lambda) = (\mu_+ - \mu_-)^2 = 1 + 4|\lambda|^2$  and the density

function is again rotationally symmetric:

$$p_{a+C_t}(\lambda) = \frac{1}{\pi t} \left( 1 - \frac{2|\lambda|^2}{(1+4|\lambda|^2)\sqrt{t^2+1+4|\lambda|^2}} + \frac{t-\sqrt{t^2+R}}{(1+4|\lambda|^2)^2} \right)$$

**Example 5.3** (Elliptic law). An interesting example is given by the so-called *elliptic random variable*  $S_\alpha + iS_\beta$ , where  $S_\alpha$  and  $S_\beta$  are free semicircular variables of variances  $\alpha$  and  $\beta$ . Note that for  $\alpha = \beta$  this is a circular variable  $C_{2\alpha}$ . The Brown measure has been computed by Haagerup (unpublished) by another method. The name *elliptic* stems from the shape of its spectrum, which is an ellipse. This can be seen as follows. Assuming that  $\alpha > \beta$  let  $\gamma = \alpha - \beta$ , then for  $\lambda \notin \sigma(S_\gamma) = [-2\sqrt{\gamma}, 2\sqrt{\gamma}]$  we have  $\lambda \in \sigma(S_\gamma + C_{2\beta})$  if and only if  $1 - C_{2\beta}(\lambda - S_\gamma)^{-1}$  is not invertible. From Theorem 2.4 we infer that the spectrum of  $C_{2\beta}(\lambda - S_\gamma)^{-1}$  is the disk centered at zero with radius  $\|C_{2\beta}(\lambda - S_\gamma)^{-1}\|_2$ , so that we get

$$\sigma(S_\gamma + C_{2\beta}) = \{\lambda : 1 \leq 2\beta \|(\lambda - S_\gamma)^{-1}\|_2^2\}$$

We use formula (3.5) for the Cauchy transform  $G_{S_\gamma}(\zeta) = \frac{\zeta - \sqrt{\zeta^2 - 4\gamma}}{2\gamma}$  to get

$$(5.4) \quad \|(\lambda - S_\gamma)^{-1}\|_2^2 = \frac{1}{2\gamma} \left( \frac{\sqrt{\lambda^2 - 4\gamma} - \sqrt{\bar{\lambda}^2 - 4\gamma}}{\lambda - \bar{\lambda}} - 1 \right)$$

and hence the spectrum is

$$\left\{ \lambda : \frac{\sqrt{\lambda^2 - 4\gamma} - \sqrt{\bar{\lambda}^2 - 4\gamma}}{\lambda - \bar{\lambda}} \geq \frac{\gamma + \beta}{\beta} = \frac{\alpha}{\beta} \right\}$$

Now consider the Zhukowski transformation  $f : \xi \mapsto \frac{1}{\xi} + \gamma\xi$ , which maps the circles  $\{\frac{e^{i\theta}}{t} : 0 \leq \theta < 2\pi\}$  to the ellipses  $\{(\frac{\gamma}{t} + t) \cos \theta + i(\frac{\gamma}{t} - t) \sin \theta : 0 \leq \theta < 2\pi\}$  and hence the open disk  $\{\xi : |\xi| < \frac{1}{\sqrt{\gamma}}\}$  bijectively onto  $\mathbf{C} \setminus [-2\sqrt{\gamma}, 2\sqrt{\gamma}]$ . Note that the excluded interval is exactly the spectrum of  $S_\gamma$ . So assume that  $\lambda = f(\xi)$  with  $|\xi| < \frac{1}{\sqrt{\gamma}}$  is not in the spectrum of  $S_\gamma$ , then observe that

$$\lambda^2 - 4\gamma = \frac{1}{\xi^2} + 2\gamma + \gamma^2\xi^2 - 4\gamma = \left( \frac{1}{\xi} - \gamma\xi \right)^2$$

and hence  $\lambda \in \sigma(S_\gamma + C_{2\beta})$  if and only if

$$\frac{\alpha}{\beta} \leq \frac{\frac{1}{\xi} - \frac{1}{\bar{\xi}} - \gamma\xi + \gamma\bar{\xi}}{\frac{1}{\xi} - \frac{1}{\bar{\xi}} + \gamma\xi - \gamma\bar{\xi}} = \frac{1 + \gamma|\xi|^2}{1 - \gamma|\xi|^2}.$$

This inequality reduces to

$$|\xi|^2 \geq \frac{1}{\alpha + \beta},$$

thus

$$\sigma(S_\gamma + C_{2\beta}) \setminus [-2\sqrt{\gamma}, 2\sqrt{\gamma}] = \left\{ f(\xi) : \frac{1}{\sqrt{\alpha + \beta}} \leq |\xi| < \frac{1}{\sqrt{\gamma}} \right\}$$

and taking the closure of this set we obtain  $\sigma(S_\gamma + C_{2\beta})$  as the interior of the ellipse

$$(5.5) \quad \left\{ \frac{2\alpha}{\sqrt{\alpha + \beta}} \cos \theta + \frac{2\beta}{\sqrt{\alpha + \beta}} i \sin \theta : 0 \leq \theta < 2\pi \right\}.$$

Now let us turn to the Brown measure. As already noted, the method from section 3.2 will not work on  $a = S_\gamma$ . Indeed the  $R$ -transform of  $|\lambda - S_\gamma|^2$  can be computed from the inverse of

$$G_{|\lambda - S_\gamma|^2}(\zeta) = \frac{1}{2\gamma} \left( 1 - \frac{\sqrt{x_+^2 - 4\gamma} - \sqrt{x_-^2 - 4\gamma}}{x_+ - x_-} \right)$$

where  $x_\pm$  are as in (3.3). Let  $\lambda = \xi + i\eta$ , then we can rewrite  $x_\pm = \xi \pm \sqrt{\zeta - \eta^2}$  and abbreviating  $y = \sqrt{\zeta - \eta^2}$ , solve the equation  $G_{|\lambda - S_\gamma|^2}(\zeta) = z$  for  $y$ , which gives

$$y^2 = \frac{\xi^2}{(1 - 2\gamma z)^2} + \frac{1}{z(1 - \gamma z)}.$$

It follows that  $K(z) = y^2 + \eta^2$  and

$$R_{|\lambda - S_\gamma|^2}(z) = z K(z) - 1 = \frac{\gamma z}{1 - \gamma z} + \frac{\xi^2 z}{(1 - 2\gamma z)^2} + \eta^2 z;$$

for real  $\lambda$  this has been used in [HKNY99] to characterize the semicircular distributions. In order to get the determining series  $f_{u|\lambda - S_\gamma|}$  according to (3.2) one has to solve a fourth order equation, which is not suitable for further computations. So we have to use formula (5.2), for which we need  $v(s)$  from (5.1) first. We have done most of the work already, since  $\int \frac{d\mu(x)}{|\lambda - x|^2 + v^2} = -G_{|\lambda - S_\gamma|^2}(-v^2)$ , thus

$$v(s)^2 = -K_{|\lambda - S_\gamma|^2}(-\frac{1}{s}) = -\left( \frac{\xi^2 s^2}{(s + 2\gamma)^2} - \frac{s^2}{s + \gamma} + \eta^2 \right)$$

and

$$\frac{v(s)^2}{s^2} = -\frac{(\lambda + \bar{\lambda})^2}{4(s + 2\gamma)^2} + \frac{1}{s + \gamma} - \frac{(\lambda - \bar{\lambda})^2}{4s^2}$$

and the density becomes, with

$$\begin{aligned} p_{S_\alpha + iS_\beta}(\lambda) &= \frac{1}{\pi} \partial_{\bar{\lambda}} \int_{t_\lambda}^{2\beta} \frac{\partial_\lambda v(s)^2}{s^2} ds \\ &= \frac{1}{\pi} \partial_{\bar{\lambda}} \int_{t_\lambda}^{2\beta} \left( -\frac{2(\lambda + \bar{\lambda})}{4(s + 2\gamma)^2} + \frac{2(\lambda - \bar{\lambda})}{4s^2} \right) ds \\ &= \frac{1}{2\pi} \partial_{\bar{\lambda}} \left( \frac{\lambda + \bar{\lambda}}{s + 2\gamma} - \frac{\lambda - \bar{\lambda}}{s} \right) \Big|_{t_\lambda}^{2\beta} \\ &= \frac{1}{4\pi} \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) - \frac{1}{2\pi} \partial_{\bar{\lambda}} \left( \frac{\lambda + \bar{\lambda}}{t_\lambda + 2\gamma} - \frac{\lambda - \bar{\lambda}}{t_\lambda} \right) \end{aligned} \tag{5.6}$$

Now  $t_\lambda = \|(\lambda - S_\gamma)^{-1}\|_2^{-2}$  has been computed above in (5.4), and denoting  $\omega = \sqrt{\lambda^2 - 4\gamma}$ , it is

$$t_\lambda = 2\gamma \left( \frac{\omega - \bar{\omega}}{\lambda - \bar{\lambda}} - 1 \right)^{-1}$$

and we claim now that the second summand in (5.6) is zero. For this note that  $\omega^2 - \bar{\omega}^2 = \lambda^2 - \bar{\lambda}^2$  and hence

$$\begin{aligned} -\frac{1}{2\pi} \partial_{\bar{\lambda}} \left( \frac{\lambda + \bar{\lambda}}{t_{\lambda} + 2\gamma} - \frac{\lambda - \bar{\lambda}}{t_{\lambda}} \right) &= -\frac{1}{4\pi\gamma} \partial_{\bar{\lambda}} \left( (\lambda + \bar{\lambda}) \left( 1 - \frac{\lambda - \bar{\lambda}}{\omega - \bar{\omega}} \right) - (\lambda - \bar{\lambda}) \left( \frac{\omega - \bar{\omega}}{\lambda - \bar{\lambda}} - 1 \right) \right) \\ &= -\frac{1}{4\pi\gamma} \partial_{\bar{\lambda}} ((\lambda + \bar{\lambda}) - (\omega + \bar{\omega}) - (\omega - \bar{\omega}) + (\lambda - \bar{\lambda})) \\ &= -\frac{1}{4\pi\gamma} \partial_{\bar{\lambda}} (2\lambda - 2\omega) \\ &= 0; \end{aligned}$$

Thus we get that the density is constant  $\frac{1}{4\pi} \left( \frac{1}{\alpha} + \frac{1}{\beta} \right)$  on the interior of the ellipse (5.5).

The elliptic law appears in the random matrix literature in [Gir97b].

## 6. OTHER EXAMPLES

There are some other examples that can be done by ad-hoc methods.

**Example 6.1.** Consider two freely independent symmetries  $u_2$  and  $v_2$  of trace zero, for example the generators of the left regular representation of  $\mathbf{Z}_2 * \mathbf{Z}_2$ . Here we compute the Brown measure of  $T = \alpha u_2 + \beta v_2$ . To get its spectrum, look at its square

$$(\alpha u_2 + \beta v_2)^2 = \alpha^2 + \beta^2 + \alpha\beta(u_2v_2 + v_2u_2)$$

Since  $u_2v_2 = (v_2u_2)^*$  is a Haar unitary, we see that  $T^2$  is a normal element with spectrum  $\sigma(T^2) = \alpha^2 + \beta^2 + \alpha\beta[-2, 2]$ . Since  $T$  and  $-T$  have the same distribution, it follows that

$$\sigma(\alpha u_2 + \beta v_2) = \left\{ \pm \sqrt{\alpha^2 + \beta^2 + \alpha\beta t} : t \in [-2, 2] \right\}$$

The Brown measure can be deduced by the same symmetry considerations, but for the sake of simplicity let us consider the special case  $\alpha = 1$ ,  $\beta = i$  only. Here the spectrum is the union of the complex intervals  $[-1 - i, 1 + i]$  and  $[-1 + i, 1 - i]$ . The Brown measure of  $(u_2 + iv_2)^2 = i(u_2v_2 + v_2u_2)$  is the arcsine law (we are taking the real part of a Haar unitary)

$$d\nu(t) = \frac{dt}{\pi\sqrt{4 - t^2}}$$

on the imaginary axis. By symmetry considerations we must have the same measure on each of the four “legs” of the spectrum, call it  $\mu_0$ , which must satisfy

$$\begin{aligned} \int_0^{\sqrt{2}} f(t^2) d\mu_0(t) &= \frac{1}{2} \int_0^2 f(t) \frac{dt}{\pi\sqrt{4 - t^2}} \\ &= \int_0^{\sqrt{2}} f(u^2) \frac{u}{\pi\sqrt{4 - u^4}} du \end{aligned}$$

and it follows that the density of the Brown measure is

$$d\mu \left( \frac{1 \pm i}{\sqrt{2}} t \right) = d\mu_0(|t|) = \frac{|t|}{\pi\sqrt{4 - t^4}} dt$$

**Example 6.2.** Other examples that are perhaps attackable arise from the following matrix models. Consider  $U_2 + A$ , where  $U_2 \in U(2N)$  is a unitary matrix s.t.  $U_2 = U_2^*$  and  $\text{tr } U_2 = 0$ , while  $A$  is an arbitrary  $2N \times 2N$  matrix. The spectrum of  $U_2 + A$  can be bounded as follows. Assume  $x$  is a unit eigenvector of  $U_2 + A$  with eigenvalue  $\lambda$ , then it can be decomposed along the

spectral projections of  $U_2$ :  $x = x_+ + x_-$  so that  $U_2x = x_+ - x_-$ . By assumption we also have  $(U_2 + A)(x_+ + x_-) = \lambda(x_+ + x_-)$ , and thus

$$x_+ = \frac{1}{2}(1 + \lambda - A)x \quad x_- = \frac{1}{2}(1 - \lambda - A)x;$$

now by orthogonality  $\langle x_+, x_- \rangle = 0$  we get

$$\begin{aligned} 0 &= \langle (1 + \lambda - A)x, (1 - \lambda + A)x \rangle \\ &= (1 + \lambda)(1 - \bar{\lambda})\|x\|^2 + (1 + \lambda)\langle x, Ax \rangle - (1 - \bar{\lambda})\langle Ax, x \rangle - \|Ax\|^2 \\ &= (1 + \lambda - \bar{\lambda} - |\lambda|^2)\|x\|^2 + (\lambda + 1)\overline{\langle Ax, x \rangle} + (\bar{\lambda} - 1)\langle Ax, x \rangle - \|Ax\|^2. \end{aligned}$$

Separating real and imaginary part results in two equations

$$\begin{aligned} 1 - |\lambda|^2 - \|Ax\|^2 + \lambda \overline{\langle Ax, x \rangle} + \bar{\lambda} \langle Ax, x \rangle &= 0 \\ \lambda - \bar{\lambda} + \overline{\langle Ax, x \rangle} - \langle Ax, x \rangle &= 0 \end{aligned}$$

Let us no consider two specific cases.

**$A$  is unitary:** In this case  $\|Ax\| = 1$  and  $\rho = \langle Ax, x \rangle$  satisfies the following equations

$$\begin{aligned} -|\lambda|^2 + \lambda\bar{\rho} + \bar{\lambda}\rho &= 0 \\ \lambda - \bar{\lambda} &= \rho - \bar{\rho} \end{aligned}$$

or in other words

$$\begin{aligned} |\lambda - \rho|^2 &= |\rho|^2 \\ \operatorname{Im} \lambda &= \operatorname{Im} \rho \end{aligned}$$

thus  $\lambda - \rho$  is real and we have

$$\lambda - \rho = \pm|\rho|$$

i.e.,

$$\lambda \in \{\rho \pm |\rho| : \rho = \langle Ax, x \rangle \in \operatorname{co} \sigma(A)\}$$

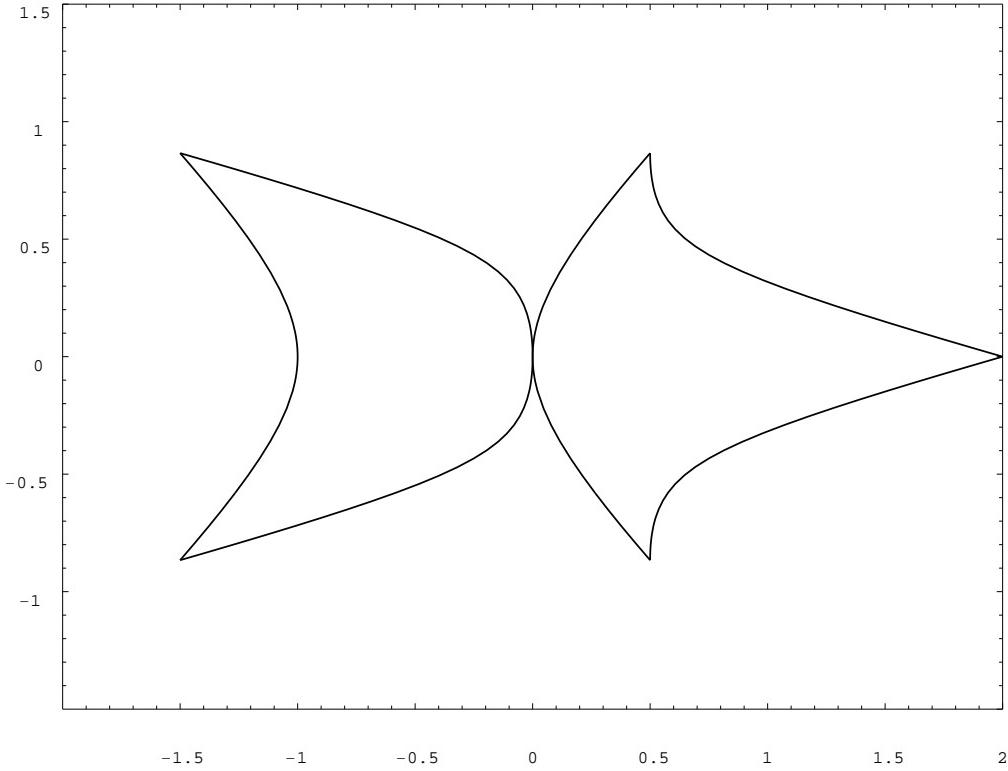
**$A = iB$  is purely imaginary:** Here we assume  $A + A^* = 0$  and the equations are

$$\begin{aligned} 1 - |\lambda|^2 - \|Bx\|^2 + i(\bar{\lambda} - \lambda)\langle Bx, x \rangle &= 0 \\ \lambda - \bar{\lambda} &= 2i \langle Bx, x \rangle \end{aligned}$$

Hence

$$\begin{aligned} \operatorname{Im} \lambda &= \langle Bx, x \rangle \\ (\operatorname{Re} \lambda)^2 &= 1 - \|Bx\|^2 + \langle Bx, x \rangle^2 \end{aligned}$$

If one puts  $A = UU_3U^*$ , where  $U_3$  is an  $6N \times 6N$  model of the generator of  $\mathbf{Z}_3$ , and  $U$  is a random unitary  $6N \times 6N$  matrix, then possible eigenvalues are enclosed by the region shown in figure 7. And indeed, samples of small numeric random unitary matrices  $U_2 + UU_3U^*$  have an eigenvalue density as shown in figure 8, while in bigger dimensions the eigenvalues concentrate, cf. figure 9. We were able to compute the spectrum of the free sum  $u_2 + u_3$  recently and will investigate this topic further in future work.

FIGURE 7. Possible spectra of random  $U_2 + U_3$ 

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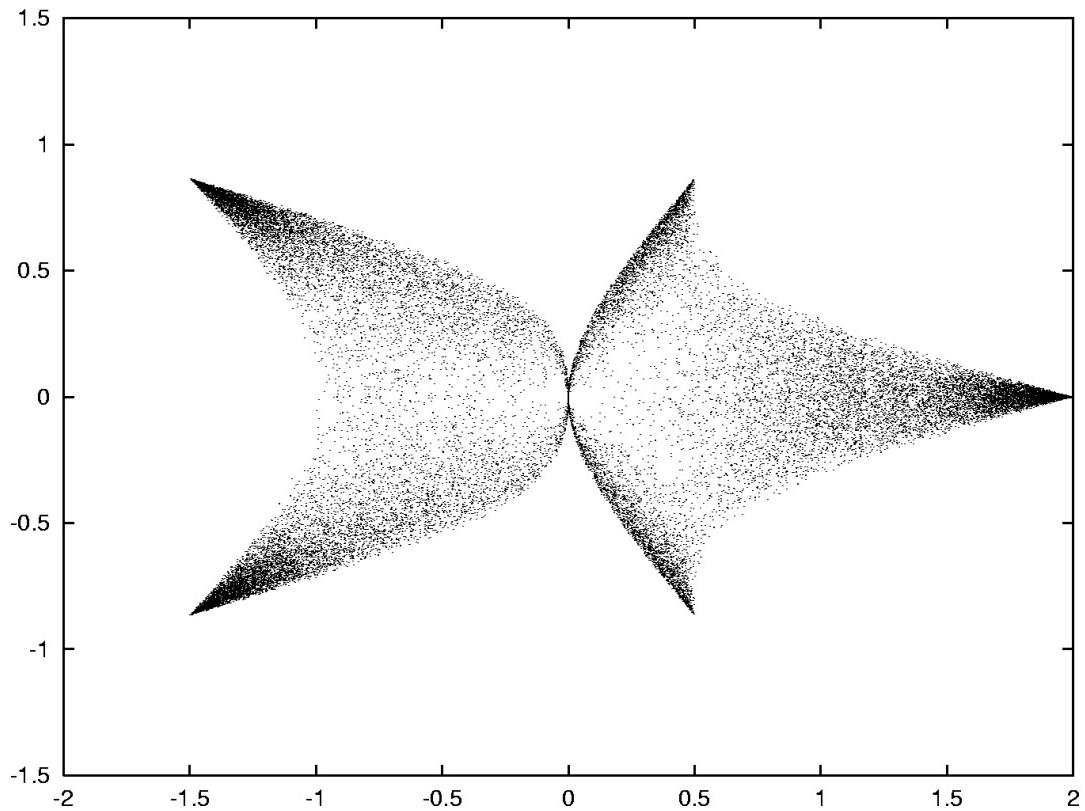


FIGURE 8. 5000 samples of eigenvalues of  $6 \times 6$  random matrices  $U_2 + U_3$

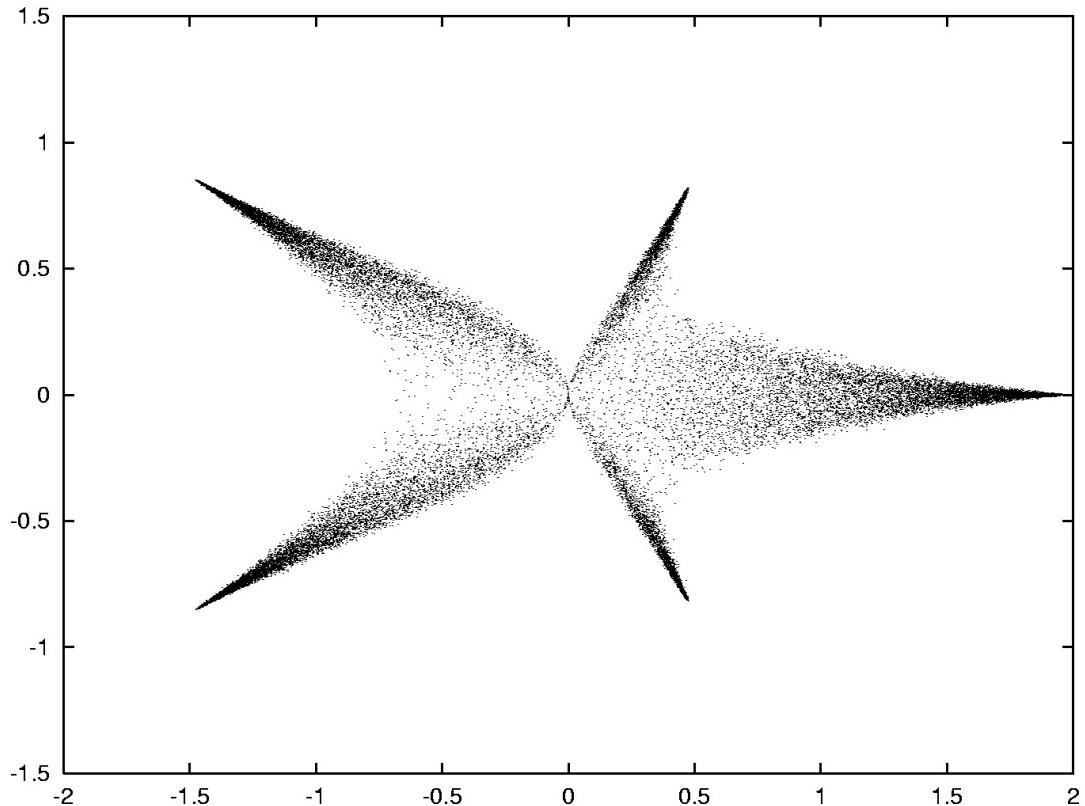


FIGURE 9. 200 samples of eigenvalues of  $150 \times 150$  random matrices  $U_2 + U_3$

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